Stochastic models of inhomogeneous complex systems: Emergence of multi-scale hierarchies

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Dedicated to my mother: Dr. Olena Ostrovska

INTRODUCTION

The sciences, economies and societies are currently experiencing paradigm shift(s) due to the unprecedented availability of complex data, ever growing amounts of computing power, and substantial advances in machine learning. In this context, a cross-disciplinary viewpoint based on the *mathematics of complex systems* emerges. In complex systems, a large number of relatively simple elements interact via the *network of interactions*. However, the collective emerging behavior cannot easily be inferred from that of the few individual components.

What are the emerging patterns in complex systems? How do they come about from the behavior of the elements? These are typical questions in the sciences, economy and policy making. These questions immediately lead to severe mathematical problems about the models of complex systems but also about their relationships with the real data.

In this thesis, we focus on several paradigmatic stochastic models of complex systems. We analyse the emergent collective behavior in inhomogeneous complex systems on multiple scales of observation, study critical phenomena, fluctuations, and attempt to explore the universality classes of these models. A common theme is the emergence of *random hierarchical structures* which describe the multi-scale behavior of the systems. Our ultimate goal is to provide fundamental and rigorous mathematical underpinnings to these phenomena.

We focus on two modeling frameworks:

- **Energy-based.** In this framework, inhomogeneous interacting components of a complex system induce a *rugged energy landscape* in which the system tries to relax to an *equilibrium*. However, the ruggedness (i.e., abundance of local extrema) can make the simple local search for an equilibrium an extremely long endeavor. This leads to important phenomena like phase transitions, ergodicity breaking, emergence of multi-scale hierarchies. While the parlance and the modeling framework originates in statistical physics (see, e.g., Bovier [43], Kadanoff [125], Ruelle [173], and Sethna [180]), it has long transcended its original realm and became important in the sciences in general Mézard *et al.* [151], e.g., computer science (see, e.g., Engel & van den Broeck [83], Mézard & Montanari [149], Opper & Saad [162], and Zdeborová & Krzakala [192]) and life sciences, see, e.g., (Gavrilets [97] and Kauffman [126]). It is thus not surprising that the energy-based models attracted substantial attention in the mathematics literature (e.g., Bovier [43], Kistler [127], Newman & Stein [157], Panchenko [164], and Talagrand [185]). Yet, despite recent spectacular advances, rigorous understanding of these phenomena remains rather limited.

- **Information-based.** A large class of complex systems can be modeled by a "population" of particles (also called "agents", depending on the context) "living" on the nodes of a network. As time progresses, the particles interact with each other by influencing each other's state (and possibly the underlying network itself) at given rates. A state represents some properties of the particle, which are of interest, e.g., an opinion or an infection status of an agent. In this setup, the dynamics of the system can be seen as exchange of information between the particles (e.g., Aldous [3] and Dawson [66]). Such *interacting* *particle systems* (IPS) is a key object of study in probability theory (see, e.g., Dawson & Greven [69], Del Moral [70], and Liggett [141]) and in the sciences (e.g., Barrat *et al.* [20], Epstein [84], Goldstone & Janssen [99], Holland [108], Nowak [159], and Porter & Gleeson [170]), where they are known under various names such as *agent-based models*. A key challenge is to understand the aggregate space-time behavior emerging in these systems.

MULTI-SCALE ANALYSIS AND EMERGENCE OF HIERARCHIES. Complex systems display emergent behavior upon increasing the scale of observation (see, e.g., Badii & Politi [16] for an introduction). A key method to analyze the structural (spatial) and dynamical aggregate behavior of stochastic models of complex systems is multi-scale analysis. Here, one considers the so-called *mesoscopic observation scales*, which lie between the micro and the macro ones. This allows to gradually pass from the microscopic to the macroscopic scale by going to increasingly larger mesoscopic scales one scale at a time.

In this thesis, the multi-scale analysis has the following two incarnations:

- For energy-based models, the multi-scale spatial structure of the systems can be discovered by tuning, e.g., the so-called *(inverse) temperature*. The conjectured picture is roughly the following. The smaller the temperature, the smaller the *essential patches* of the configuration space on which the system *concentrates*. Vice versa, by increasing the temperature, the essential patches coalesce into larger essential patches. This way, by varying the temperature, the whole hierarchy of the essential patches (also called *pure states*) emerges.[1]

- A key approach to understand *universality* and to study the emergent spatio-temporal patterns in the behavior of interacting systems is renormalization analysis, e.g., Greven [100] and Kadanoff [125]. Here, mesoscopic observables are analyzed on an increasing sequence of observation scales. In an ideal situation, this analysis results in a *renormalization mapping* between the emergent patterns on two consecutive scales of observation. In this case, by iterating the renormalization mappings, one obtains the orbit of patterns emerging in the increasing sequence of scales of observation. The *universality classes* can then be associated with the attractors of such orbits.

INHOMOGENEITIES. In this thesis, there is an additional explicit source of multi-scale behavior in the models we focus on: the presence of the inhomogeneous background, which is modeled by a random environment. The models which we consider have at least two a priori scales: slow degrees of freedom (representing the inhomogeneous background/random environment), and the fast degrees of freedom (representing the foreground).

ORIGINAL PUBLICATIONS. This habilitation thesis is based on the following original publications:

1a. A. Klimovsky. Towards renormalisation of the hierarchically interacting heavy-tailed Lambda-Cannings models. *In:* The Renormalization Group (Eds. M. Disertori, J. Feldman and M. Salmhofer), *Oberwolfach Reports*, **8** (1), 2011.

1 This picture is supposed to have profound implications on the dynamics in energy based models. In particular, dynamics in rugged landscapes displays rich features: the epochs of (transient) stasis are punctuated by "paradigm shifts". We do not pursue dynamics questions for energy-based models in this thesis.

2a. A. Klimovsky. High-dimensional Gaussian fields with isotropic increments seen through spin glasses. *Electron. Commun. Probab.*, **17**, 1–14, 2012.

3a. A. Greven, F. den Hollander, S. Kliem, A. Klimovsky. Renormalisation of hierarchically interacting Lambda-Cannings processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, **11** (1), 43–140, 2014.

4a. Z. Kabluchko and A. Klimovsky. Complex random energy model: Zeros and fluctuations. *Prob. Theor. and Rel. Fields*, **158**, 159–196, 2014.

5a. L. Hartung, A. Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.* **20** (78), 2015.

6a. L. Hartung, A. Klimovsky. The phase diagram of the complex temperature branching Brownian motion energy model. *Electron. J. Probab.* **23** (127), 2018.

7a. A. Greven, F. den Hollander, A. Klimovsky. The hierarchical Cannings process in random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, **15**, 295–351, 2018.

8a. J. Černý, A. Klimovsky. Markovian dynamics of exchangeable arrays. *In:* Genealogies of Interacting Particle Systems (Eds. M. Birkner, R. Sun, J. Swart), Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, Volume 38, World Scientific, 2020.

9a. Z. Kabluchko and A. Klimovsky. Generalized random energy model at complex temperatures. *Submitted*, 109 pp., 2019. Available at http://arxiv.org/abs/1402.2142

ORGANIZATION. In this thesis, we present the main results from the above listed original publications. We mostly do not include the proofs. However, we try to give some heuristic explanations/hints on why the results are plausible. Furthermore, we provide additional context, relationships, extensions, open problems and possible avenues for future research.

The remainder of the thesis is organized as follows:

- **Part I** is devoted to energy-based models of disordered systems. Informally, a *disordered system* is a complex system with configurations/states "living" on (at least) two levels/scales: the "background" (or slow) level – representing the state of the inhomogeneous (disordered) background environment, and the "foreground" (or fast) level – representing the state of the system *given* the state of the background. There are two components to an energy based model – a *state space* and an *energy function* on this state space. In disordered systems, the energy function is moreover a *stochastic process*, which models the inhomogeneous background. Each realization of the stochastic process gives the state of the background.

  The first part contains four chapters. We focus on four energy based models with increasingly more complicated stochastic processes of energies:

  - **Chapter 1** focuses on the **random energy model** (REM). The REM is probably the simplest but arguably paradigmatic model of an energy-based disordered system, which displays a *phase transition*. The energy process in this model is a white noise and the state space is a finite set without any additional structure. While the original model is well understood, our main contribution is the analysis of a complex-valued extension of the model. The motivation for the complex-valued setup is multi-fold: (1) the *Lee-Yang theory* of phase-transitions which identifies phase transitions with the accumulation points of the complex-plane zeros of the partition function; (2) *quantum physics* and *interference phenomena;* (3) important mathematics objects such as the *Riemann zeta function*, *characteristic polynomials of large random matrices* and distributions of their zeros. For the complex-valued REM, we identify the stochastic fluctuations of the partition function, derive the phase diagram and study the distribution of complex plane zeros of the so-called partition function. The value of this model is not only pedagogical. It comes with a universality class and has features, which are shared with much more complicated models, some of which can be witnessed in Chapters 2, 3 and 4.

This chapter is based on publication 4a.

– **Chapter 2** is devoted to the analysis of the the **generalized random energy model** (GREM) at complex temperatures. This model is an successful and arguably informative attempt to go beyond the REM universality class. In this model, the stochastic process of energies is a strongly correlated Gaussian process with correlations given by a (function of) the *genealogical distance* on the deterministic tree with a fixed depth but of growing breadth. This tree structure induces *multi-scale* stochastic fluctuations and produces a rich phase diagram, which hopefully sheds some light on the complex plane phase diagrams of models with more complicated energy processes. As with the complex-valued REM, the crucial step is the analysis of the fluctuations of the partition function. This allows for detailed results on the distribution and fluctuations of the zeros of the partition function.

  This chapter is based on publication 9a.

– **Chapter 3** provides the phase diagram and fluctuations in the complex **branching Brownian motion (BBM) energy model**. *Branching Brownian motion* plays the rôle of the energy process in this model. Due to the its branching structure, as GREM, BBM has correlations given by the genealogical distance on a tree. However, the underlying tree is given by a *Galton-Watson* tree, which is a random tree of growing depth and breadth. It turns out that this model lies exactly at the borderline of the REM universality class. It has the same complex plane phase diagram as the complex REM but the stochastic fluctuations are different from those of the REM due to the strong correlations of the energy process.

  This chapter is based on publications 5a., 6a.

– **Chapter 4** studies rather generic **high-dimensional Gaussian fields with isotropic increments** playing the rôle of the energy function. We study the phase diagram of this model without directly resorting to the analysis of its fluctuations, as it was done in Chapters 1-3. We use very different methods from those of the first three chapters. Our methods rely on: (1) the perturbative analysis of the model (so-called *Aizenman-Sims-Starr* scheme); (2) comparison with suitably chosen scaling limits of the GREM (so-called *Ruelle's probability cascades*); (3) stochastic symmetries (exchangeability, Ghirlanda-Guerra identities and *Panchenko*'s proof of ultrametricity).

This chapter is based on publication 2a.

- **Part II** is devoted to information-based models. We consider space-time models with Markovian dynamics on high-dimensional state spaces. Specifically, we focus on interacting particle systems on networks. The main challenge is to identify the aggregate behavior emerging in these systems.

  This part contains two chapters:

  - **Chapter 5** introduces and studies the **spatial Cannings model in random environment**. This is an extension of a central stochastic model for multi-type population dynamics called the *Cannings model*. We consider an evolving population of multi-type particles in a hierarchically structured geographical space subject to the following dynamics: reproduction under constrained amount of resources (resampling), non-local catastrophes (correlated updates affecting the whole blocks of the geographical space) and migration of individuals in the geographical space. In this model, somewhat similarly to Part I, the mechanisms of reproduction and that of catastrophes are assumed to be *inhomogeneous* across the geographical space. The inhomogeneities and are modeled by a *random environment*. To model the evolution of a multi-type population, we use the framework of *measure-valued Markov processes*. In this framework, the measures represent the empirical distribution of types at given spatial location. To study the large space-time scale behavior of the model, we employ the methods of: (1) multi-scale analysis/renormalization group ideas; (2) duality methods from the theory of interacting particle systems, which relate the behavior of IPS with the behavior of simpler stochastic processes.

    This chapter is based on publications 1a., 3a., 7a.

  - **Chapter 6** introduces and studies a model of **evolving networks**. This model is one of the first steps towards studying large-scale stochastic limiting objects in *interacting particle systems on evolving networks*. This is an emerging research topic in probability theory and statistics with a plethora of applications in the sciences. We study how some popular assumptions like *Markovian evolution*, *vertex exchangeability* and *subsampling consistency* imposed on a large weighted graph play together. In the large network limit, we study the limiting object for such networks. This object can be seen as a *Markovian dynamics of exchangeable two-dimensional arrays*. We classify the possible jumps of such dynamics. At a jump time, one of the tree transitions occurs: "microscopic" (a single entry in the array jumps); "mesoscopic" (a proportion of a row/column elements jumps); or "macroscopic" (a proportion of the whole two dimensional array jumps). We answer the question "Are the subsamples of an exchangeable Markov array also Markov?" in the negative and provide a counterexample. This is because there can exist non-local exchangeable quantities in the Markovian array. We show, however, that under additional continuity assumptions on the Markov semigroup (the so-called Feller property), any subsample of a Feller exchangeable array is indeed Markov.

    This chapter is based on publication 8a.

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Part I

ENERGY-BASED MODELS

1 COMPLEX RANDOM ENERGY MODEL

What causes phase transitions in energy based disordered systems? How do they come about?
Motivated by these questions, but also by quantum physics, interference phenomena and by mathematical objects such as the Riemann zeta-function and characteristic polynomials of large random matrices, we consider the *complex random energy model*.
We study the stochastic fluctuations in this model. We identify the asymptotic structure of *complex zeros* of the so-called partition function. This structure provides an explanation for the phase transitions. We discuss how correlations between the real and imaginary parts of the random energy influence the fluctuations and the phase diagram.
The value of this model is not only pedagogical: it comes with a universality class and has features which are shared with much more complicated models. Some of these we consider in Chapters 2, 3 and 4.

This chapter is based on publication 4a.

1.1 INTRODUCTION

Random energy model is a paradigmatic (and particularly simple) model of a disordered system which displays a *phase transition*.

ENERGY BASED MODELS AND PHASE TRANSITIONS. An energy based model is a model defined by a configuration/state space and an energy function on it. Given an *energy function* $\{H_n(s)\}_{s\in S_n}$ on the configuration space S_n of of *size* $n \in \mathbb{N}$, a classical manifestation of **phase transitions** is the *loss of analyticity* of the *log-partition function* $p_n(\beta)$:

$$p_n(\beta) := \frac{1}{n} \log \mathcal{Z}_n(\beta), \quad \beta \in \mathbb{R}_+ \tag{1.1}$$

with

$$\mathcal{Z}_n(\beta) := \sum_{s\in S_n} \exp(\beta H_n(s)), \quad \beta \in \mathbb{R}_+, \tag{1.2}$$

as the size of the system increases $n \to \infty$ (i.e., in the so-called *thermodynamic limit*). Obviously, we have to make suitable choices of the configuration space, of the energy function and of the notion of size, so that the thermodynamic limit makes sense and is non-trivial.

RANDOM ENERGY MODEL. The REM was introduced by Derrida [71, 72]. In the REM, the configuration space has very little structure:

$$S_N := \{1, \dots, N\}, \quad N \in \mathbb{N}. \tag{1.3}$$

where N is the *number of configurations* of the system. The energy function in the REM is random. It is a *stochastic process* and a very simple one: *white noise*. In the subsequent chapters, we consider more involved/realistic stochastic processes playing the rôle of the energy function. Nevertheless, the simple model we are starting here with will serve as a guidance.

Let $X, X_1, X_2, \dots$ be independent real standard normal random variables of which we think of as of energies assigned to the configurations. The *partition function* of the REM at inverse temperature β is defined by

$$\mathcal{Z}_N(\beta) = \sum_{k=1}^{N} e^{\beta\sqrt{n}X_k}, \tag{1.4}$$

where we use the notation $n = \log N$.

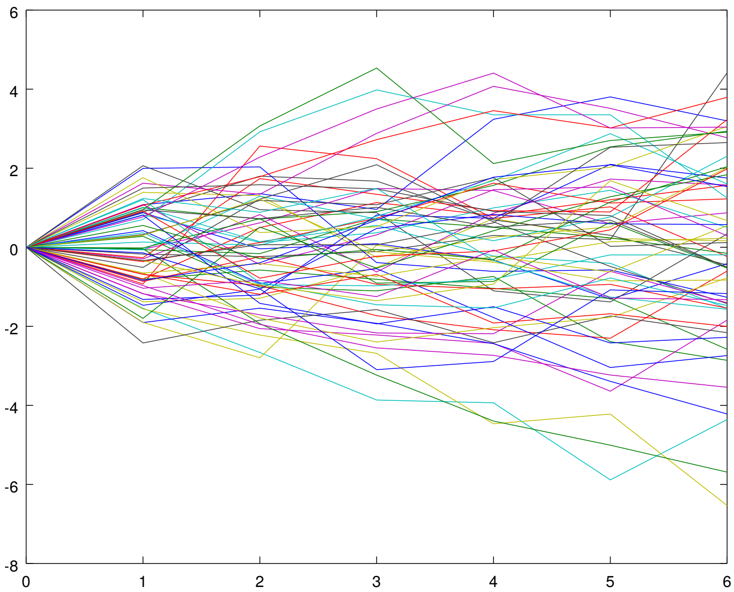


Figure 1.1: Simulation of the REM field of energies X generated as positions of the independent random walkers (with Gaussian increments) at the right edge of the plot. The "random walk perspective" is chosen to compare the REM to models with strong correlations from the subsequent chapters, cf. Figures 2.2 and 3.1. The idea for the plot is adapted from Ouimet [163]

HEURISTICS. Why $\sqrt{n}$ scaling in (1.4)? In line with a statistical physics convention, we would like the logarithm of the partition function to be an *extensive quantity* in n, i.e.,

$$\log \mathcal{Z}_N(\beta) \underset{N\to\infty}{\sim} p(\beta)n \quad \text{with high probability}, \tag{1.5}$$

where $p(\beta)$ is an n-independent constant, cf. (1.1). For (1.5) to hold, in view of the trivial bounds

$$\frac{1}{n}\log\left(\max_{k=1}^{N} e^{\beta\sqrt{n}X_k}\right) \le \frac{1}{n}\log \mathcal{Z}_N(\beta) \le \frac{1}{n}\log\left(N\max_{k=1}^{N} e^{\beta\sqrt{n}X_k}\right) = 1 + \frac{1}{n}\log\left(\max_{k=1}^{N} e^{\beta\sqrt{n}X_k}\right), \tag{1.6}$$

it is necessary that the maximal summand $\max_{k=1}^{N} e^{\beta\sqrt{n}X_k}$ in the partition function (1.4) is of order $e^{M(\beta)n}$ for some n-independent constant $M(\beta)$ with high probability. It is a standard fact from extreme value theory (see, e.g., Leadbetter *et al.* [137, Example 1.7.1]) that

$$\max_{k=1}^{N} X_k \underset{N\to\infty}{\sim} \sqrt{2n}, \quad \text{a.s.} \tag{1.7}$$

This implies the desired order of the maximal term with $M(\beta) = \sqrt{2}\beta$. See also Section 2.5, and (2.19) in particular, for further probabilistic heuristics on the asymptotic behavior of partition functions.

A PHASE TRANSITION. For *real* inverse temperatures $\beta > 0$, Derrida [71, 72] heuristically studied the asymptotic behavior of $\log \mathcal{Z}_N(\beta)$ as $N \to \infty$ (or equivalently, as $n \to \infty$) and computed that

$$p(\beta) := \lim_{N\to\infty} \frac{1}{n}\log \mathcal{Z}_N(\beta) = \begin{cases} 1 + \frac{1}{2}\beta^2, & 0 \le \beta \le \sqrt{2}, \\ \sqrt{2}\beta, & \beta \ge \sqrt{2}. \end{cases} \tag{1.8}$$

This formula was confirmed using rigorous probabilistic arguments establishing that convergence (1.8) holds both a.s. and in L^q, $q \ge 1$; see Bovier [43], Eisele [82], and Olivieri & Picco [161].

An immediate consequence of (1.8) is that $p(\beta)$ is non-analytic at $\beta_c = \sqrt{2}$ which is according to the physics nomenclature (cf., e.g., Ruelle [173]) a manifestation of a *phase transition* at β_c.

From the analytic viewpoint, phase transitions can be understood within the realm of complex analysis by considering $\beta \in \mathbb{C}$ as we discuss in the next paragraph.

THE LEE-YANG PROGRAM. Since the pioneering work of Lee and Yang [138, 191], much attention in the statistical physics literature has been paid to studying partition functions of various models at *complex* values of parameters such as complex inverse temperature or complex external magnetic field, see, e.g., Bena *et al.* [25] and Biskup *et al.* [33]. These studies are sometimes referred to as the Lee–Yang program. The motivation here is to identify the mechanisms causing phase transitions of the model under study. These transitions manifest themselves in the analyticity breaking of the logarithm of the partition function which, in turn, is related to the *complex zeros* of the partition function. Phase transitions are thus associated with the accumulation points of the complex zeros of the partition function on the real axis, in the large system limit. In this respect, complex-valued parameters provide a clean framework for identification of phase transitions. The main emphasis of the Lee–Yang program was on the classical lattice models of statistical mechanics.

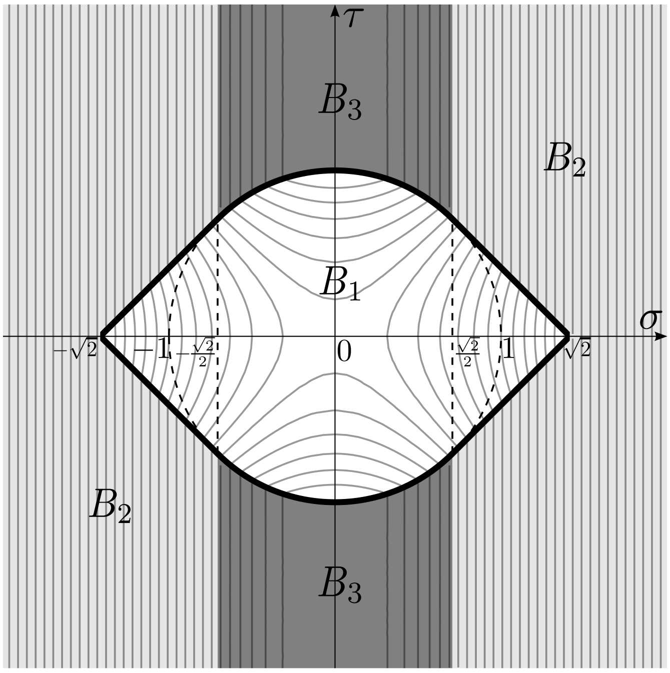


Figure 1.2: Complex zeros of $\mathcal{Z}_N$ in the large N limit. There are three phases: B_1 (white, no zeros), B_2 (light gray, density of zeros is of order 1), B_3 (dark gray, density of zeros is of order n). On the boundary of B_1 the linear density of zeros is of order n. The plot shows also the contour lines (gray curves and lines) of the log-partition function p.

Using heuristic arguments, Derrida [74] studied the REM at *complex* inverse temperature $\beta = \sigma + i\tau$. He derived the following logarithmic asymptotics extending (1.8) to the complex plane:

$$p(\beta) := \lim_{N\to\infty} \frac{1}{n} \log |\mathcal{Z}_N(\beta)| = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3, \end{cases} \tag{1.9}$$

where B_1, B_2, B_3 are three subsets of the complex plane (see Figure 1.2) defined by

$$B_1 = \mathbb{C} \backslash \overline{B_2 \cup B_3}, \tag{1.10}$$

$$B_2 = \{\beta \in \mathbb{R}^2 \colon 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \tag{1.11}$$

$$B_3 = \{\beta \in \mathbb{R}^2 \colon 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \tag{1.12}$$

Here, $\bar{A}$ denotes the closure of the set A. Note that the limiting log-partition function p is continuous.

To derive (1.9), Derrida [74] used an approach which can be roughly described as follows. Instead of $\mathcal{Z}_N(\beta)$, one can consider the truncated sum

$$\mathcal{Z}_N^*(\beta) = \sum_{k=1}^{N} \mathrm{e}^{\beta\sqrt{n}X_k} \mathbb{1}_{|X_k| < \sqrt{2n}}.$$

Indeed, with high probability it holds that $\mathcal{Z}_N^*(\beta) = \mathcal{Z}_N(\beta)$ since the order of the maximum term among $|X_1|, \dots, |X_N|$ is $\sqrt{2n}$ and the existence of an outlier satisfying $|X_k| > \sqrt{2n}$ has probability converging to 0. Note, however, that although $\mathcal{Z}_N^*(\beta)$ and $\mathcal{Z}_N(\beta)$ are close in probability, their expectations (and standard deviations) may be very different from each other, at least for some values of β. Derrida derived an asymptotic formula for the expectation of $\mathcal{Z}_N^*(\beta)$, as $N \to \infty$, using the saddle-point method. Two cases are possible: the expectation is dominated by the energies X_k inside the interval $(-\sqrt{2n}, \sqrt{2n})$ (equivalently, the contribution of the saddle point dominates the expectation), or by the energies located near one of the boundary points $\pm\sqrt{2n}$. He also obtained two similar cases for the standard deviation of $\mathcal{Z}_N^*(\beta)$. Comparing the resulting four formulas, Derrida discovered the three phases B_1, B_2, B_3. The arguments of Derrida [74] are not fully rigorous, although it should be emphasized that he did not use the replica method or other standard non-rigorous spin glass method. In this chapter, we make the argument of Derrida rigorous and refine his results by deriving distributional limit theorems for the fluctuations of $\mathcal{Z}_N(\beta)$ (and for the fluctuations in some more general models, see Section 1.4) at complex β. An essential feature of the REM at complex temperature is the possibility of canceling of terms in $\mathcal{Z}_N(\beta)$ due to the presence of complex amplitudes. It is for this reason that some standard techniques of rigorous spin glass theory [185] like the concentration inequalities or the second-moment method do not (or do not always) lead to the desired result.

Based on his formula (1.9) for the limiting log-partition function, Derrida [74] computed the asymptotic distribution of zeros of $\mathcal{Z}_N$ in the complex plane. His predictions were in a good agreement with the numerical simulations of Moukarzel & Parga [153]. Derrida observed that since $\mathcal{Z}_N(\beta)$ is an analytic function of β, its empirical distribution of zeros (a measure Ξ_N assigning to every zero of $\mathcal{Z}_N$ a weight equal to its multiplicity) is given by

$$\Xi_N = \frac{1}{2\pi}\Delta \log|\mathcal{Z}_N|, \tag{1.13}$$

where $\Delta = \frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}$ denotes the Laplace operator in the β-plane. In fact, identity (1.13) should rigorously be understood in the sense of distributions (= generalized functions), cf. Remark 1.3.1. Taking the large N limit, Derrida obtained the formula $\frac{n}{2\pi}\Delta p$ for the asymptotic distribution of zeros of $\mathcal{Z}_N$. Since the function p is harmonic in B_1 and B_2, Derrida predicted that "there should be no zeros (or at least that the density of zeros vanishes) in phases B_1 and B_2". In phase B_3, "the density of zeros is uniform" and is asymptotic to $\frac{n}{2\pi}$. Also, since the normal derivative of p has a jump on the boundary of B_1, but has no jump on the boundary between B_1 and B_3 "the boundaries between phases B_1 and B_2, and between phases B_1 and B_3 are lines of zeros whereas the separation between phases B_2 and B_3 is not". The argument of Derrida involves interchanging the Laplace operator and the large N limit. In the this chapter, we justify Derrida's approach rigorously and derive further results on the distribution of zeros of $\mathcal{Z}_N$. Namely, we relate the zeros of $\mathcal{Z}_N$ to the zeros of two random analytic functions: a Gaussian analytic function $\mathbb{G}$ (in phase B_3), and a zeta-function ζ_P associated to the Poisson process (in phase B_2). Also, we will clarify the local structure of the mysterious "lines of zeros" on the boundary of B_1.

For the partition function of REM, considered as a function of a complex external magnetic field, a non-rigorous analysis similar to that of Derrida [74] has been carried out by Moukarzel & Parga [154, 155]. For directed polymers with complex weights on a tree, which is another related model, the logarithmic asymptotics (1.9) has been derived in [75]; see also [27, 131]. Recently, Takahashi [183] and Obuchi & Takahashi [160] studied the complex zeros in the generalized REM and other spin glass models using the non-rigorous replica method. However, spin glasses at complex temperature have not been much studied rigorously in the mathematics literature. Our aim is to fill this gap.

SUMMARY OF MOTIVATIONS. The motivation to consider the complex-valued setup is multi-fold:

1. **Critical phenomena.** Lee and Yang [191] observed that *phase transitions* (= analyticity breaking of the log-partition function) occur at critical points due to the accumulation of *complex zeros* of the partition function (viewed as a function of the external field) around the critical points on the real line, as the size of the system tends to infinity (= thermodynamic limit).

2. **Quantum physics and interference phenomena.** The formalism of quantum physics is based on the sums (and integrals) of *complex exponentials*. This naturally leads to cancellations between the magnitudes of the summands in the partition function. This is a manifestation of the interference phenomenon, see, e.g., Derrida *et al.* [75] and Dobrinevski *et al.* [78]. See also the studies of the quantum Monte Carlo method Düring & Kurchan [80].

3. **Random analytic functions.** The sum of random exponentials $\mathcal{Z}_N$ is a natural random analytic function exhibiting, despite of its simple form, a rather non-trivial behavior. We hope that the methods developed to study this function can be applied to other random analytic functions, for example to random polynomials or random Taylor series. For a recent work in this direction, we refer to Kabluchko [122] and Kabluchko & Zaporozhets [124]. Also, $\mathcal{Z}_N$ can be interpreted as a (normalized) characteristic function of the i.i.d. normal sample $X_1, \dots, X_N$. This connection will be discussed in Section 1.6.

4. **Random matrix theory and the Riemann zeta function.** The Riemann zeta function is a central object of analytic number theory. Striking relationships between statistical physics of random energy models and randomized versions of the zeta function and characteristic polynomials of random matrices were conjectured by Fyodorov et al. [95].

1.2 NOTATION

We will write the complex inverse temperature β in the form $\beta = \sigma + i\tau$, where $\sigma, \tau \in \mathbb{R}$. We use the notation $n = \log N$, where N is a large integer and the logarithm is natural. Note that in the physics literature on the REM, it is customary to take the logarithm at basis 2. Replacing β by $\beta/\sqrt{\log 2}$ in our results we can easily switch to the physics notation.

We denote by $N_{\mathbb{R}}(0, s^2)$ the real Gaussian distribution with mean zero and variance $s^2 > 0$. By $N_{\mathbb{C}}(0, s^2)$, we denote the complex Gaussian distribution with density

$$z \mapsto \frac{1}{\pi s^2} e^{-|z/s|^2}$$

w.r.t. the Lebesgue measure on $\mathbb{C}$. Note that $Z \sim N_{\mathbb{C}}(0, s^2)$ iff $Z = X + iY$, where $X, Y \sim N_{\mathbb{R}}(0, s^2/2)$ are independent. In this case, $\mathbb{E}Z = 0$ and $\mathbb{E}|Z|^2 = 1$. Real or complex normal distribution is referred to as standard if $s = 1$. The standard normal distribution function is denoted by Φ.

Convergence in probability and weak (distributional) convergence will be denoted by $\xrightarrow{P}$ and $\xrightarrow{w}$, respectively. Let C be a generic positive constant whose value will change at different occurrences.

1.3 RESULTS ON ZEROS

Let $\mathcal{Z}_N$ be the partition function of the REM defined as in (1.4). Note the distributional equalities

$$\mathcal{Z}_N(\beta) \stackrel{d}{=} \mathcal{Z}_N(-\beta), \quad \mathcal{Z}_N(\bar{\beta}) \stackrel{d}{=} \overline{\mathcal{Z}_N(\beta)}. \tag{1.14}$$

Due to (1.14), it is often enough to consider the case $\sigma, \tau \geq 0$. The next result describes the global structure of complex zeros of $\mathcal{Z}_N$, as $N \to \infty$. Let Ξ_3 be the Lebesgue measure restricted to B_3. Also, let Ξ_{13} be the one-dimensional length measure on the boundary between B_1 and B_3 (which consists of two circular arcs). Finally, let Ξ_{12} be a measure having the density $\sqrt{2}|\tau|$ with respect to the one-dimensional length measure restricted to the boundary between B_1 and B_2 (which consists of four line segments). Define a measure $\Xi = 2\Xi_3 + \Xi_{12} + \Xi_{13}$.

Theorem 1.3.1. *For every continuous function $f : \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\frac{1}{n} \sum_{\beta \in \mathbb{C}:\ \mathcal{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{P} \frac{1}{2\pi} \int_{\mathbb{C}} f(\beta)\Xi(\mathrm{d}\beta). \tag{1.15}$$

Remark 1.3.1. *As a consequence, the random measure assigning a weight $1/n$ to each zero of $\mathcal{Z}_N$ converges weakly to the deterministic measure $\frac{1}{2\pi}\Xi$. The limit measure Ξ is related to the limiting log-partition function p, see (1.9), by the formula $\Xi = \Delta p$, in accordance with [74]. Here, Δ is the Laplace operator which should be understood in the distributional sense. The point-wise Laplacian of p is easily seen to be $2\Xi_3$. However, in the distributional Laplacian there are additional terms which come from the jumps of the normal derivative of p along the boundaries $\bar{B}_1 \cap \bar{B}_2$ and $\bar{B}_1 \cap \bar{B}_3$. On the boundary $\bar{B}_2 \cap \bar{B}_3$ the jump turns out to be 0. that p can be viewed as the two-dimensional electrostatic potential generated by the charge distribution Ξ.*

Theorem 1.3.1 makes the last formula in [74] rigorous. In the next theorems, we will investigate more fine properties of the zeros of $\mathcal{Z}_N$. We start by describing the local structure of zeros of $\mathcal{Z}_N$ in a neighborhood of area $1/n$ of a fixed point $\beta_0 \in B_3$. Let $\{\mathbb{G}(t) : t \in \mathbb{C}\}$ be a Gaussian random analytic function [156] given by

$$\mathbb{G}(t) = \sum_{k=0}^{\infty} \xi_k \frac{t^k}{\sqrt{k!}}, \tag{1.16}$$

where $\xi_0, \xi_1, \ldots$ are independent standard complex Gaussian random variables. The complex zeros of $\mathbb{G}$ form a remarkable point process which has intensity $1/\pi$ and is translation invariant. Up to rescaling, this is the only translation invariant zero set of a Gaussian analytic function; see [111, Section 2.5]. This and related zero sets have been much studied; see the monograph [111].

Theorem 1.3.2. *Let $\beta_0 \in B_3$ be fixed. For every continuous function $f\colon \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\sum_{\beta\in\mathbb{C}\colon \mathcal{Z}_N(\beta)=0} f(\sqrt{n}(\beta-\beta_0)) \xrightarrow[N\to\infty]{w} \sum_{\beta\in\mathbb{C}\colon \mathbb{G}(\beta)=0} f(\beta).$$

Remark 1.3.2. *Equivalently, the point process consisting of the points $\sqrt{n}(\beta-\beta_0)$, where β is a zero of $\mathcal{Z}_N$, converges weakly to the point process of zeros of $\mathbb{G}$.*

Derrida [74] predicted that the set B_1 should be free of zeros. As we will see below, it is not true that the number of zeros in B_1 converges to 0 in probability since with non-vanishing probability there exist zeros very close to the boundary of B_1. However, a slightly weaker statement is true.

Theorem 1.3.3. *Let K be a compact subset of B_1. Then, there exists $\varepsilon > 0$ depending on K such that*

$$\mathbb{P}[\mathcal{Z}_N(\beta)=0, \textit{ for some } \beta\in K] = O(N^{-\varepsilon}), \quad N\to\infty.$$

As a consequence, the number of zeros of $\mathcal{Z}_N$ in K converges to 0 in probability. It is natural to conjecture that the convergence holds a.s. The number ε, as provided by the proof of Theorem 1.3.3, converges to 0 as the distance between K and the boundary of B_1 gets smaller. So, the a.s. convergence does not follow from a Borel–Cantelli argument.

Consider now the zeros of $\mathcal{Z}_N$ in the set B_2. We will show that in the limit as $N\to\infty$ the zeros of $\mathcal{Z}_N$ in B_2 look like the zeros of certain random analytic function ζ_P. This function may be viewed as a zeta-function associated to the Poisson process. It is defined as follows. Let $P_1 < P_2 < \ldots$ be the arrival times of a unit intensity homogeneous Poisson process on the positive half-line. That is, $P_k = \varepsilon_1 + \ldots + \varepsilon_k$, where $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. standard exponential random variables, i.e., $\mathbb{P}[\varepsilon_k > t] = \mathrm{e}^{-t}$, $t\ge 0$. For $T>1$, define the random process

$$\tilde{\zeta}_P(\beta;T) = \sum_{k=1}^{\infty} \frac{1}{P_k^{\beta}} \mathbb{1}_{P_k\in[0,T]} - \int_1^T t^{-\beta}\mathrm{d}t, \quad \beta\in\mathbb{C}. \tag{1.17}$$

Theorem 1.3.4. *With probability 1, the sequence $\tilde{\zeta}_P(\beta;T)$ converges as $T\to\infty$ to a limiting function denoted by $\tilde{\zeta}_P(\beta)$. The convergence is uniform on compact subsets of the half-plane $\{\beta\in\mathbb{C}\colon \operatorname{Re}\beta > 1/2\}$.*

Corollary 1.3.1. *With probability 1, the Poisson process zeta-function*

$$\zeta_P(\beta) = \sum_{k=1}^{\infty} \frac{1}{P_k^{\beta}} \tag{1.18}$$

defined originally for $\operatorname{Re}\beta > 1$, admits a meromorphic continuation to the domain $\operatorname{Re}\beta > 1/2$. The function $\tilde{\zeta}_P(\beta) = \zeta_P(\beta) - \frac{1}{\beta-1}$ is a.s. analytic in this domain.

The next theorem describes the limiting structure of zeros of $\mathcal{Z}_N$ in B_2. The form of the process ζ_P appearing there is not surprising and can be explained as follows. In phase B_2 the process $\mathcal{Z}_N$ is dominated by the extremal order statistics of the sample $X_1,\ldots,X_N$. These form a Poisson point process in the large N limit, see, e.g., [171, Corollary 4.19(i)], and ζ_P is some functional of this process.

Theorem 1.3.5. *Let $f\colon B_2 \to \mathbb{R}$ be a continuous function with compact support. Let $\zeta_P^{(1)}$ and $\zeta_P^{(2)}$ be two independent copies of ζ_P. Then,*

$$\sum_{\beta\in B_2\colon\, \mathcal{Z}_N(\beta)=0} f(\beta) \xrightarrow[N\to\infty]{w} \sum_{\substack{\beta\in B_2\colon\\ \zeta_P^{(1)}(\beta/\sqrt{2})=0}} f(\beta) + \sum_{\substack{\beta\in B_2\colon\\ \zeta_P^{(2)}(\beta/\sqrt{2})=0}} f(-\beta). \tag{1.19}$$

Theorem 1.3.5 tells us that the zeros of $\mathcal{Z}_N$ in the domain $\sigma > 1/\sqrt{2}, |\sigma| + |\tau| > \sqrt{2}$ (which constitutes one half of B_2) have approximately the same law as the zeros of ζ_P, as $N \to \infty$. Let us stress that the approximation breaks down in the triangle $\sigma > 1/\sqrt{2}$, $|\sigma| + |\tau| < \sqrt{2}$. Although the function ζ_P is well-defined and may have zeros there, the function $\mathcal{Z}_N$ has, with high probability, no zeros in any compact subset of the triangle by Theorem 1.3.3.

Next, we state some properties of the function ζ_P. Let $\beta > 1/2$ be real. For $\beta \neq 1$, the random variable $\zeta_P(\beta)$ is stable with index $1/\beta$ and skewness parameter 1. In fact, (1.17) is just the series representation of this random variable; see [178, Theorem 1.4.5]. For $\beta = 1$, the random variable $\tilde{\zeta}_P(1)$ (which is the residue of ζ_P at 1) is 1-stable with skewness 1. For general complex β, we have the following stability property.

Proposition 1.3.1. *If $\zeta_P^{(1)}, \ldots, \zeta_P^{(k)}$ are independent copies of ζ_P, then we have the following distributional equality of stochastic processes:*

$$\zeta_P^{(1)} + \ldots + \zeta_P^{(k)} \stackrel{d}{=} k^\beta \zeta_P. \tag{1.20}$$

To see this, observe that the union of k independent unit intensity Poisson processes has the same law as a single unit intensity Poisson process scaled by the factor $1/k$. As a corollary, the distribution of the random vector $(\operatorname{Re}\zeta_P(\beta), \operatorname{Im}\zeta_P(\beta))$ belongs to the family of operator stable laws; see [147].

Proposition 1.3.2. *Fix $\tau \in \mathbb{R}$. As $\sigma \downarrow 1/2$, we have*

$$\sqrt{2\sigma - 1}\,\zeta_P(\sigma + i\tau) \xrightarrow{w} \begin{cases} N_{\mathbb{C}}(0,1), & \text{if } \tau \neq 0, \\ N_{\mathbb{R}}(0,1), & \text{if } \tau = 0. \end{cases} \tag{1.21}$$

As a corollary, there is a.s. no meromorphic continuation of ζ_P beyond the line $\sigma = 1/2$. Using the same method of proof, it can be shown that for every different $\tau_1, \tau_2 > 0$ the random variables $\sqrt{2\sigma - 1}\,\zeta_P(\sigma + i\tau_j)$, $j = 1, 2$, become asymptotically independent as $\sigma \downarrow 1/2$. Thus, the function ζ_P looks like a naïve white noise near the line $\sigma = 1/2$. The intensity of complex zeros of ζ_P at β can be computed by the formula $g(\beta) = \frac{1}{2\pi}\Delta\mathbb{E}\log|\zeta_P(\beta)|$, where Δ is the Laplace operator; see [111, Section 2.4]. Proposition 1.3.2 suggests that $g(\beta) \sim \frac{1}{\pi}\frac{1}{(2\sigma-1)^2}$ as $\sigma \downarrow 1/2$. In particular, every point of the line $\sigma = 1/2$ should be an accumulation point for the zeros of ζ_P with probability 1.

Let us look locally at the zeros of $\mathcal{Z}_N$ near some $\beta_0 = \sigma_0 + i\tau_0$ on one of the boundaries $\bar{B}_1 \cap \bar{B}_3$ or $\bar{B}_1 \cap \bar{B}_2$. We will show that in both cases the zeros form approximately an arithmetic sequence. The structure of the measures Ξ_{13} and Ξ_{12} in Theorem 1.3.1 suggests that the distances between the consequent zeros should behave like $\frac{2\pi}{n}$ in the first case and like $\frac{\sqrt{2}\pi}{|\tau_0|n}$ in the second case. The next theorems show that this is indeed true. First, we analyze the boundary $\bar{B}_1 \cap \bar{B}_3$.

Theorem 1.3.6. *Let $\beta_0 = \sigma_0 + i\tau_0$ be such that $\sigma_0^2 + \tau_0^2 = 1$ and $\sigma_0^2 < 1/2$. There exist a complex-valued random variable ξ and a bounded real sequence δ_N such that for every continuous function $f\colon \mathbb{C} \to \mathbb{R}$ with compact support,*

$$\sum_{\beta\in\mathbb{C}\colon \mathcal{Z}_N(\beta)=0} f\left(n\left(\frac{\beta}{\beta_0}-1\right)-i\delta_N\right) \overset{w}{\underset{N\to\infty}{\longrightarrow}} \sum_{k\in\mathbb{Z}} f(2\pi ik+\xi). \tag{1.22}$$

Remark 1.3.3. *In other words, the zeros of $\mathcal{Z}_N$ near β_0 are given by the formula*

$$\beta = \beta_0\left(1+\frac{2\pi ik+\xi+i\delta_N}{n}\right)+o\left(\frac{1}{n}\right), \quad k\in\mathbb{Z}. \tag{1.23}$$

As we will see in the proof, the random variable $\operatorname{Re}\xi$ takes negative values with positive probability. It follows that the probability that $\mathcal{Z}_N$ has a zero in B_1 does not go to 0 as $N\to\infty$.

The boundary $\bar{B}_1 \cap \bar{B}_2$ consists of 4 line segments. By symmetry (1.14), it suffices to consider one of them.

Theorem 1.3.7. *Let $\beta_0 = \sigma_0 + i\tau_0$ be such that $\sigma_0 > 1/\sqrt{2}$, $\tau_0 > 0$ and $\sigma_0 + \tau_0 = \sqrt{2}$. There exist a complex-valued random variable η and a complex sequence $d_N = O(\log n)$ such that for every continuous function $f\colon \mathbb{C}\to\mathbb{R}$ with compact support,*

$$\sum_{\beta\in\mathbb{C}\colon \mathcal{Z}_N(\beta)=0} f\left(e^{\frac{2\pi i}{3}} n(\beta-\beta_0)-d_N\right) \overset{w}{\underset{N\to\infty}{\longrightarrow}} \sum_{k\in\mathbb{Z}} f\left(\frac{2\pi ik+\eta}{\sqrt{2}\tau_0}\right). \tag{1.24}$$

Remark 1.3.4. *In other words, the zeros of $\mathcal{Z}_N$ near β_0 are given by the formula*

$$\beta = \beta_0 + e^{-\frac{2\pi i}{3}}\frac{1}{n}\left(\frac{2\pi ik}{\sqrt{2}\tau_0}+d_N\right)+o\left(\frac{1}{n}\right), \quad k\in\mathbb{Z}. \tag{1.25}$$

1.4 RESULTS ON FLUCTUATIONS

We state our results on fluctuations for a generalization of (1.4) which we call complex random energy model. This model involves complex phases and allows for arbitrary dependence between the energies and the phases. Let $(X,Y),(X_1,Y_1),\ldots$ be i.i.d. zero-mean bivariate Gaussian random vectors with

$$\operatorname{Var}X_k = \operatorname{Var}Y_k = 1, \quad \operatorname{Corr}(X_k,Y_k)=\rho. \tag{1.26}$$

Here, $-1\le\rho\le 1$ is fixed. Recall (1.4) and consider the following partition function:

$$\mathcal{Z}_N(\beta) = \sum_{k=1}^{N} e^{\sqrt{n}(\sigma X_k + i\tau Y_k)}, \qquad \beta=(\sigma,\tau)\in\mathbb{R}^2. \tag{1.27}$$

For $\tau=0$, this is the REM of Derrida [72] at real inverse temperature σ. For $\rho=1$, we obtain the REM at the complex inverse temperature $\beta=\sigma+i\tau$ considered above; see (1.4). For $\rho=0$, the model is a REM with independent complex phases considered in [75]. Note also that the substitutions $(\beta,\rho)\mapsto(-\beta,\rho)$ and $(\beta,\rho)\mapsto(\bar{\beta},-\rho)$ leave the distribution of $\mathcal{Z}_N(\beta)$ unchanged.

Recall (1.8). Define the *log-partition function* as

$$p_N(\beta) = \frac{1}{n}\log|\mathcal{Z}_N(\beta)|, \qquad \beta=(\sigma,\tau)\in\mathbb{R}^2. \tag{1.28}$$

Theorem 1.4.1. *For every $\beta \in \mathbb{R}^2$, the limit*

$$p(\beta) := \lim_{N\to\infty} p_N(\beta) \tag{1.29}$$

exists in probability and in L^q, $q \geq 1$, and is explicitly given as

$$p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B}_1, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B}_2, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B}_3. \end{cases} \tag{1.30}$$

Note that the limit in (1.30) does not depend on ρ. However, we will see below that the fluctuations of $\mathcal{Z}_N(\beta)$ do depend on ρ. The next theorem shows that $\mathcal{Z}_N(\beta)$ satisfies the central limit theorem in the domain $\sigma^2 < 1/2$.

Theorem 1.4.2. *If $\sigma^2 < 1/2$ and $\tau \neq 0$, then*

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau\rho}}{N^{\frac{1}{2}+\sigma^2}} \xrightarrow[N\to\infty]{w} N_{\mathbb{C}}(0,1). \tag{1.31}$$

Remark 1.4.1. *If $\sigma^2 < 1/2$ and $\tau = 0$, then the limiting distribution is real normal, as was shown in [42].*

Remark 1.4.2. *If in addition to $\sigma^2 < 1/2$ we have $\sigma^2 + \tau^2 > 1$, then $N^{1+\frac{1}{2}(\sigma^2-\tau^2)} = o(N^{\frac{1}{2}+\sigma^2})$ and, hence, the theorem simplifies to*

$$\frac{\mathcal{Z}_N(\beta)}{N^{\frac{1}{2}+\sigma^2}} \xrightarrow[N\to\infty]{w} N_{\mathbb{C}}(0,1). \tag{1.32}$$

Eq. (1.32) explains the difference between phases B_1 and B_3: in phase B_1 the expectation of $\mathcal{Z}_N(\beta)$ is of larger order than the mean square deviation, whereas, in phase B_3, vice versa: the mean square deviation is larger than the expectation. It is this behavior that leads to the phase transition between B_1 and B_3 in (1.30).

In the boundary case $\sigma^2 = 1/2$, the limiting distribution is normal, but it has truncated variance.

Theorem 1.4.3. *If $\sigma^2 = 1/2$ and $\tau \neq 0$, then*

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\frac{1}{2}-\tau^2)+i\sigma\tau\rho}}{N} \xrightarrow[N\to\infty]{w} N_{\mathbb{C}}(0,1/2).$$

Next, we describe the fluctuations of $\mathcal{Z}_N(\beta)$ in the domain $\sigma^2 > 1/2$. Due to (1.14), it is not a restriction of generality to assume that $\sigma > 0$. Let b_N be a sequence such that $\sqrt{2\pi} b_N e^{b_N^2/2} \sim N$ as $N \to \infty$. We can take

$$b_N = \sqrt{2n} - \frac{\log(4\pi n)}{2\sqrt{2n}}. \tag{1.33}$$

Theorem 1.4.4. *Let $\sigma > 1/\sqrt{2}$, $\tau \neq 0$, and $|\rho| < 1$. Then,*

$$\frac{\mathcal{Z}_N(\beta) - N\mathbb{E}[e^{\sqrt{n}(\sigma X + i\tau Y)}\mathbb{1}_{X<b_N}]}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma}, \tag{1.34}$$

where S_α denotes a complex isotropic α-stable random variable with a characteristic function of the form $\mathbb{E}[e^{i\,\mathrm{Re}(S_\alpha \bar{z})}] = e^{-\mathrm{const}\cdot|z|^\alpha}$, $z \in \mathbb{C}$.

Remark 1.4.3. *If $\sigma > 1/\sqrt{2}$ and $\tau = 0$, then the limiting distribution is real totally skewed α-stable; see [42]. If $\sigma > 1/\sqrt{2}$ and $\rho = 1$ (resp., $\rho = -1$), then can be shown that that*

$$\frac{\mathcal{Z}_N(\beta) - N\mathbb{E}[e^{\beta\sqrt{n}X}\mathbb{1}_{X<b_N}]}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \tilde{\zeta}_P\left(\frac{\beta}{\sqrt{2}}\right) \quad \left(\text{resp., } \tilde{\zeta}_P\left(\frac{\bar{\beta}}{\sqrt{2}}\right)\right). \tag{1.35}$$

Remark 1.4.4. *The truncated expectation on the left-hand side of (1.34) can be computed. It can be shown that under the assumptions of Theorem 1.4.4,*

$$\frac{\mathcal{Z}_N(\beta)}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma}, \quad \text{if } \sigma + |\tau| > \sqrt{2}, \tag{1.36}$$

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau\rho}}{e^{\sigma\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} S_{\sqrt{2}/\sigma}, \quad \text{if } \sigma + |\tau| \leq \sqrt{2}. \tag{1.37}$$

Similarly, if $\sigma > 1/\sqrt{2}$, but $\rho = 1$, then we have

$$\frac{\mathcal{Z}_N(\beta)}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \zeta_P\left(\frac{\beta}{\sqrt{2}}\right), \quad \text{if } \sigma + |\tau| > \sqrt{2}, \tag{1.38}$$

$$\frac{\mathcal{Z}_N(\beta) - N^{1+\frac{1}{2}(\sigma^2-\tau^2)+i\sigma\tau}}{e^{\beta\sqrt{n}b_N}} \xrightarrow[N\to\infty]{w} \zeta_P\left(\frac{\beta}{\sqrt{2}}\right), \quad \text{if } \sigma + |\tau| \leq \sqrt{2},\ \sigma \neq \sqrt{2}. \tag{1.39}$$

For $\rho = -1$, we have to replace β by $\bar{\beta}$.

1.5 RELATED RESULTS

LEE-YANG PROGRAM. The Lee-Yang approach [138, 191] to phase transitions (cf. Section 2.3) is a part of standard books on mathematical Statistical Physics for many decades [173]. We refer to the works Biskup et al. [31, 32], Borcea and Brändén [36], Fröhlich and Rodriguez [90] for the mathematical state of the art of this program and further references[1]. Shamis & Zeitouni [181] studied the classical Curie-Weiss model at complex temperatures The above works, however, do not concern with *disordered systems* which are the main focus of this work.

Several models of of complex-valued random energy landscapes were considered in the literature. We group them according to the strength of correlations.

1 Bena et al. [25] provides a review of the physics literature. Interestingly, even experimental measurements of the partition function zeros via quantum interference are possible, see Peng et al. [168].

INDEPENDENT ENERGIES. The REM was suggested by Derrida [71] as a simple spin glass model and independently by Lifshitz *et al.* [140] and Pastur [167] as a model for wave propagation in inhomogeneous media. Though simple, this model plays a paradigmatic rôle in the physics of disordered systems, see, e.g., Fyodorov & Bouchaud [94] and references therein. Mathematically, the behaviour of the REM at real inverse temperature is well understood; see Bovier *et al.* [42] and Bovier [43, Chapter 9]. For the REM at *complex* inverse temperature, Derrida [74] derived the limiting free energy, obtained the phase diagram and computed the limiting distribution of complex zeros of the partition function. Moukarzel and Parga confirmed Derrida's results numerically Moukarzel & Parga [153] and studied REM in the complex external field Moukarzel & Parga [154, 155]. Koukiou Koukiou [131] studied analyticity of the partition function of the REM in a complex plain neighborhood of the origin.

In [123], the results of Derrida [74] were confirmed rigorously via the probabilistic analysis of *fluctuations* of the partition function. As a consequence of the fluctuation results, it was shown in [123] that the limiting log-partition function is given by (3.11) and does not depend on the *correlation parameter* ρ, cf. (3.6).

LOGARITHMIC CORRELATIONS. A class of models with the so-called logarithmic correlations turns out to be exactly at the borderline of the REM universality class in the following sense: the phase diagram is the same as in the REM, however the limiting fluctuations of the partition function are already different from the case of independent energies (REM).

In [75], Derrida et al. considered a landscape of complex-valued random energies attached to the leaves of a deterministic regular tree of fixed depth, as the depth tends to infinity. Similarly to the locations of the BBM particles, the energies of the leaves are generated as a sum of the independent complex-valued weights collected along the path connecting the root to a leaf. This can be seen as a *mean-field model of directed polymers with random complex weights on the regular tree*. For this model, under the assumption $\rho = 0$, the authors of [75] showed that the very same formula (3.11) holds for the directed polymer without resorting to the more informative analysis of fluctuations of the partition function.

Barral et al. [18, 19] studied *complex Gaussian multiplicative cascades* on the unit interval. These works cover Phase I (cf. Fig. 3.2) via a martingale convergence result. In Phase II, the authors show tightness of the properly rescaled partition function. The model is constructed using a dyadic embedding of the binary tree into the unit interval. This makes the model closely related to that of [75].

On Euclidean spaces in higher dimensions ($d \geq 2$), under the assumption $\rho = 0$, a random energy model on Euclidean spaces with logarithmic (w.r.t. the Euclidean distance) correlations was studied by Lacoin et al. [133]. In [133], for this *Gaussian multiplicative chaos*, the same phase diagram as on Figure 3.2 was identified. However, only Phases I and III were treated in [133]. Maduale et al. [144] studied the complex cascade model on a regular binary tree closely related to the models of [18, 19, 75]. On the boundary between Phases I and II, [144] provides a modulus of continuity estimate for the chaos. For a review on Gaussian multiplicative chaos, we refer to Rhodes and Vargas [172]. Purely imaginary multiplicative chaos was studied by Junnila et al. [121].

Phase II was studied for the *complex branching BBM energy model* in the case $\rho = 0$ in [144]. We analyze this model with $\rho \neq 0$ in Chapter 3.

As mentioned in the remark below Theorem 3.5.2, the branching structure of the BBM implies complex distributional equations (3.25), which are referred to as *complex smoothing transform*. A detailed study on how solutions to such equations with complex weights look like was recently done by Meiners and Mentemeier [148], see also the recent paper by Kolesko and Meiners [129]. The case of real-valued scalar weights was treated by Alsmeier and Meiners [7] and by Iksanov and Meiners [115].

Fluctuations of the so-called additive (Biggins') martingale (which is nothing else as the partition function) for a supercritical branching random walk were studied for complex temperatures by Iksanov et al. [114]. In the real-valued case, fluctuations of the derivative martingale for the BBM (cf., (3.19)) were identified by Maillard and Pain [146].

Hairer and Shen [106] studied the *dynamical sine-Gordon model* – a non-linear parabolic SPDE in two spatial dimensions subject to additive space-time white noise. In [106], it is shown that the corresponding Hairer's regularity structure is related to the complex multiplicative Gaussian chaos from [133].

Striking conjectures on the relationships of the log-correlated (complex) random energy models with *characteristic polynomials of random matrices*, and the *Riemann zeta function* were formulated by Fyodorov et al. [95]. Some of these conjectures have been considered in the mathematics literature, see, e.g., Arguin et al. [9, 11], and Saksman and Webb [177].

GOING DEEP BEYOND REM: GREM. What happens beyond the above mentioned borderline of logarithmic correlations of the REM universality class? Motivated by Parisi's theory of the hierarchical organization of the pure states and related hierarchical *replica symmetry breaking* for the Sherrington-Kirkpatrick (SK) model [151], Derrida introduced the GREM; see [73, 76, 77]. This model has "designed" hierarchical correlations. Rigorous results on the GREM at real inverse temperatures were obtained by Capocaccia et al. [50] and in a series of works by Bovier and Kurkova [39–41]. For a review of these results, we refer to Bovier and Kurkova [45] and Bovier [43, Chapter 10]. The recent progress in rigorous understanding of the SK-type models draws heavily on the analysis of fluctuations in the GREM and their relation to the SK model; see [164]. Using the non-rigorous replica method, Takahashi [183] computed the log-partition function of the GREM at complex temperatures. In Chapter 2, we rigorously confirm and extend the results of [183].

LEE-YANG PROGRAM. The Lee-Yang approach [138, 191] to phase transitions (cf. Section 2.3) is a part of standard books on mathematical Statistical Physics for many decades [173]. We refer to the works Biskup et al. [31, 32], Borcea and Brändén [36], Fröhlich and Rodriguez [90] for the mathematical state of the art of this program and further references[2]. Shamis & Zeitouni [181] studied the classical Curie-Weiss model at complex temperatures The above works, however, do not concern with *disordered systems* which are the main focus of this work.

2 Bena et al. [25] provides a review of the physics literature. Interestingly, even experimental measurements of the partition function zeros via quantum interference are possible, see Peng et al. [168].

1.6 DISCUSSION, EXTENSIONS AND OPEN QUESTIONS

The results on fluctuations are closely related, at least on the heuristic level, to the results on the zeros of $\mathcal{Z}_N$. In Section 1.4, we claimed that regardless of the value of $\beta \neq 0$ we can find normalizing constants $m_N(\beta) \in \mathbb{C}$, $v_N(\beta) > 0$ such that

$$\frac{\mathcal{Z}_N(\beta) - m_N(\beta)}{v_N(\beta)} \xrightarrow[N\to\infty]{w} Z(\beta)$$

for some non-degenerate random variable $Z(\beta)$. It turns out that in phase B_1 the sequence $m_N(\beta)$ is of larger order than $v_N(\beta)$, which suggests that there should be no zeros in this phase. In phases B_2 and B_3, the sequence $m_N(\beta)$ is of smaller order than $v_N(\beta)$, which does not rule out the possibility of zeros in these phases. One way to guess the density of zeros in phases B_2 and B_3 is to look more closely at the correlations of the process $\mathcal{Z}_N$. In phase B_3, it can be shown that $\mathcal{Z}_N(\beta_1)$ and $\mathcal{Z}_N(\beta_2)$ become asymptotically decorrelated if the distance between β_1 and β_2 is of order larger than $1/\sqrt{n}$. This suggests that the distances between the close zeros in phase B_3 should be of order $1/\sqrt{n}$ and hence, the density of zeros should be of order n. Similarly, in phase B_2 the variables $\mathcal{Z}_N(\beta_1)$ and $\mathcal{Z}_N(\beta_2)$ remain non-trivially correlated at distances of order 1, which suggests that the density of zeros in this phase should be of order 1.

An additional motivation for studying $\mathcal{Z}_N$ comes from its connection to the empirical characteristic function. Given an i.i.d. standard normal sample $X_1, \dots, X_N$, the empirical characteristic function is defined by $c_N(\beta) = \sum_{k=1}^{N} e^{i\beta X_k}$. We have $\mathcal{Z}_N(\beta) = c_N(-i\sqrt{n}\beta)$. The limit behavior of the stochastic process $\{c_N(\beta)\colon \beta \in \mathbb{R}\}$ without rescaling β by the factor $\sqrt{n}$ has been much studied; see, e.g., [62, 88]. There has been also interest in the behavior of $R_N = \inf\{\beta > 0\colon \operatorname{Re} c_N(\beta) = 0\}$, the first real zero of $\operatorname{Re} c_N$; see [107, 113]. In particular, it has been shown in [107, Corollary 4.5] that, for all $t \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}[R_N^2 - n < 2t] = \Phi(-\sqrt{2}e^{-t}).$$

Hence, the first real zero of $\operatorname{Re} \mathcal{Z}_N(\beta)$ restricted to $\beta \in i\mathbb{R}$ is located near i with high probability. This is exactly the point where the imaginary axis meets the set B_3.

It is possible to extend or strengthen our results in several directions. The statements of Theorem 1.3.1 and Theorem 1.4.1 should hold almost surely, although it seems difficult to prove this. Several authors considered models involving sums of random exponentials generalizing the REM; see [24, 35, 60, 120]. They analyze the case of real β only. We believe that our results (both on zeros and on fluctuations) can be extended, with appropriate modifications, to these models.

2 COMPLEX GENERALIZED RANDOM ENERGY MODEL

Using the so-called *Generalized Random Energy Model at complex temperatures*, we explore the phase diagrams of disordered systems with *complex energies* beyond the universality class of the REM from Chapter 1. We identify the complete phase diagram of the GREM in the complex plane and describe the global limiting distribution of complex zeros of the partition function. We show that depending on $\beta \in \mathbb{C}$ the random energy of each level of the GREM contributes to the free energy in one of three possible ways: via the extremal values (glassy phase), via the variance (fluctuation phase) or via the mean value (expectation phase). Given a GREM with d levels, it turns out that there are $\frac{1}{2}(d+1)(d+2)$ phases in total, each of which can symbolically be encoded as $G^{d_1}F^{d_2}E^{d_3}$ with $d_1, d_2, d_3 \in \mathbb{N}_0$ such that $d_1 + d_2 + d_3 = d$, where d is the depth of the GREM tree. The encoding is deciphered as follows: in phase $G^{d_1}F^{d_2}E^{d_3}$, the first d_1 levels (counting from the root of the GREM tree) are in the *glassy* phase (G), the next d_2 levels are in the *fluctuation* phase (F), and the last d_3 levels are in the *expectation* phase (E). Moreover, we identify the limiting fluctuations of the partition function. We show that if $|\operatorname{Re}\beta|$ is small enough, then the limiting fluctuations of the partition function are *Gaussian*, otherwise the fluctuations are *non-Gaussian* and are influenced by *Poisson cascades* of extremal energies. This knowledge of fluctuations immediately gives the local distributions of zeros of the partition function viewed as a function of β. As a corollary, we get an explicit formula for the free energy and derive the phase diagram. Finally, we discuss implications of the above results and a number of conjectures on disordered systems with complex energies.

This chapter is based on publication 9a.

What happens beyond the above mentioned (see Section 1.5) borderline of logarithmic correlations of the REM universality class? Motivated by Parisi's theory of the hierarchical organization of the pure states and related hierarchical *replica symmetry breaking* for the Sherrington-Kirkpatrick (SK) model Mézard *et al.* [151], Derrida introduced the GREM; see [73, 76, 77]. This model has "designed" hierarchical correlations. Rigorous results on the GREM at real inverse temperatures were obtained by Capocaccia et al. [50] and in a series of works by Bovier & Kurkova [39–41]. For a review of these results, we refer to Bovier & Kurkova [45] and Bovier [43, Chapter 10]. The recent progress in rigorous understanding of the SK-type models draws on the analysis of fluctuations in the GREM and their relation to the SK model; see Panchenko [164] and Ruelle [174]. Using the non-rigorous replica method, Takahashi [183] computed the log-partition function of the GREM at complex temperatures. In this chapter, we rigorously confirm and extend the results of Takahashi [183].

2.1 THE MODEL

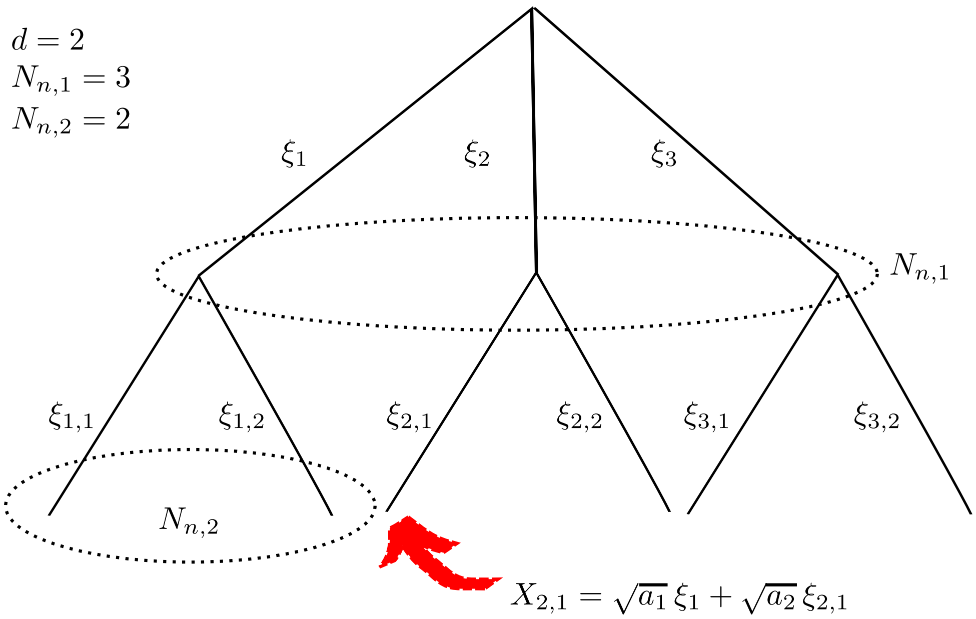


Figure 2.1: Sketch of the GREM tree with $d = 2$ levels and independent standard Gaussian random variables ξ attached to the edges of the tree.

Following Derrida [73], given the *tree depth* $d \in \mathbb{N}$ and the parameter $n \in \mathbb{N}$ describing the model size, consider the set of the *tree leaves*

$$\mathbb{S}_n := \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{N}^d : 1 \le \varepsilon_1 \le N_{n,1}, \dots, 1 \le \varepsilon_d \le N_{n,d}\}. \tag{2.1}$$

In the tree underlying $\mathbb{S}_n$, the *branching numbers* $N_{n,j}$ (possibly) depend on the *tree level* $j \in [d] := \{1, \dots, d\}$. We assume that $N_{n,j}$ grow exponentially in n, so that, for some given constants $\alpha_j > 1$, $j \in [d]$,

$$\lim_{n\to\infty} \frac{N_{n,j}}{\alpha_j^n} = 1, \quad \alpha_j > 1, \quad j \in [d]. \tag{2.2}$$

The GREM is based on the zero-mean Gaussian random field $X = \{X_\varepsilon : \varepsilon \in \mathbb{S}_n\}$ indexed by $\mathbb{S}_n$ and given by

$$X_\varepsilon := \sqrt{a_1}\,\xi_{\varepsilon_1} + \sqrt{a_2}\,\xi_{\varepsilon_1\varepsilon_2} + \dots + \sqrt{a_d}\,\xi_{\varepsilon_1\dots\varepsilon_d}, \quad \varepsilon \in \mathbb{S}_n \tag{2.3}$$

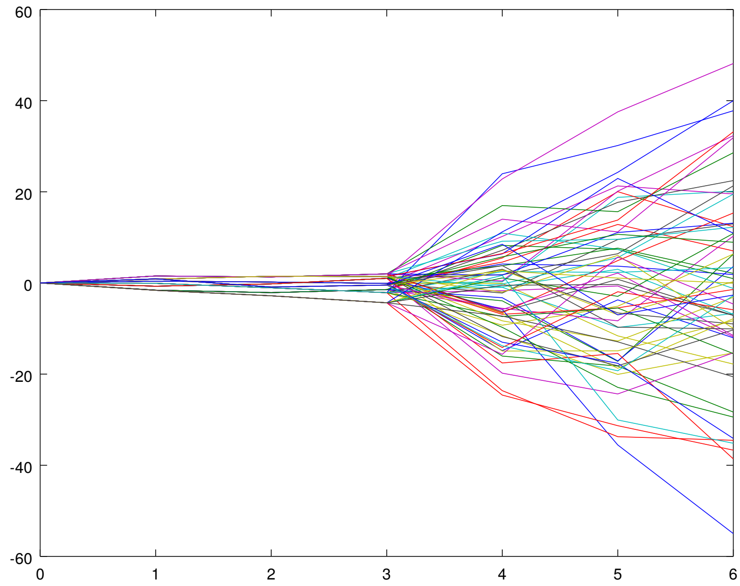


Figure 2.2: Simulation of the GREM field of energies X via random walks branching at the same deterministic point in time (here $t = 3$, $d = 2$, $a_2 > a_1$). The field is given by the positions of the random walkers at the right edge of the plot. Compare Figures 1.1 and 3.1. The idea for the plot is adapted from Ouimet [163]

in terms of independent real standard normal random variables

$$\xi := \{\xi_{\varepsilon_1 \ldots \varepsilon_j} \colon 1 \le \varepsilon_1 \le N_{n,1}, \ldots, 1 \le \varepsilon_j \le N_{n,j}, \quad j \in [d]\}. \tag{2.4}$$

We can think of $\xi_{\varepsilon_1 \ldots \varepsilon_j}$ as of attached to the edge $\varepsilon_1 \ldots \varepsilon_j$ of the tree at the level $j \in [d]$[1], see Figure 2.1.

2.2 PHASE TRANSITIONS

Under the assumption that the constants $\{\sigma_j\}_{j=0}^{d+1}$ given by

$$\sigma_0 := 0, \quad \sigma_j := \sqrt{\frac{2 \log \alpha_j}{a_j}}, \quad 1 \le j \le d, \quad \sigma_{d+1} := +\infty \tag{2.5}$$

form an increasing sequence[2], i.e.,

$$\sigma_1 < \ldots < \sigma_d, \tag{2.6}$$

1 So, X_ε is the sum of the independent Gaussian random variables $\xi_{\varepsilon_1,\ldots,\varepsilon_j}$ with weights $\sqrt{a_j}$, $j \in [d]$ attached to the edges along the path connecting the leaf $\varepsilon \in \mathbb{S}_n$ with the root of the tree.

2 In fact, (2.6) is a convexity assumption on the broken line connecting the points with abscissæ 0, $a_1 + \ldots + a_j$ and ordinates 0, $\log \alpha_1 + \ldots + \log \alpha_j$, $j \in [d]$ respectively. It is natural to consider the GREM even without this assumption, which was done in the seminal work of Bovier and Kurkova [40, 41]. It should be more or less straightforward to extend the results of the present work to such setups. We are not concerned with this extension here.

Derrida and Gardner [77] heuristically derived the following formula for the limiting *log-partition function* for the GREM as $n \to \infty$

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\epsilon \in \mathbb{S}_n} \exp\left(\beta\sqrt{n}X_\epsilon\right) = \beta \sum_{k=1}^{m} \sqrt{2a_k \log \alpha_k} + \sum_{k=m+1}^{d} \left(\log \alpha_k + \frac{1}{2} a_k \beta^2\right), \quad \text{a.s.} \tag{2.7}$$

where the *inverse temperature* parameter $\beta \in [\sigma_m, \sigma_{m+1})$, $m \in \{0, \ldots, d\}$. This formula was rigorously proved by Capocaccia et al. [50]; see Bovier and Kurkova [40] for much more. Clearly, (2.7) implies that $p(\cdot)$ is *non-analytic* in any neighborhood of the set $\{\sigma_j\}_{j=1}^d$. However, for finite $n \in \mathbb{N}$, the log-partition function

$$p_n(\beta) := \frac{1}{n} \log \sum_{\epsilon \in \mathbb{S}_n} \exp\left(\beta\sqrt{n}X_\epsilon\right), \quad \beta \in \mathbb{R} \tag{2.8}$$

is analytic on $\mathbb{R}$. Therefore, the points $\{\sigma_j\}_{j=1}^d$ are called the *critical inverse temperatures* or the *points of phase transition*.

Question 2.2.1. *Why does the analyticity breaking (= phase transition) occur in the log partition function (cf. (2.7)) as $n \to \infty$?*

2.3 LEE-YANG APPROACH TO PHASE TRANSITIONS

A classical way to answer Question 2.2.1 was suggested by Lee and Yang [138, 191] and Fisher [89]: Show that $\{\sigma_j\}_{j=1}^d$ are the accumulation points as $n \to \infty$ of the complex plane *zeros* of the *partition function*

$$\mathcal{Z}_n(\beta) := \sum_{\varepsilon \in \mathbb{S}_n} e^{\beta\sqrt{n}X_\varepsilon}, \quad \beta = \sigma + i\tau \in \mathbb{C}, \quad \sigma, \tau \in \mathbb{R}. \tag{2.9}$$

Therefore, even though the critical points $\{\sigma_j\}_{j=1}^d \subset \mathbb{R}$ themselves are never zeros of the partition function for any $n \in \mathbb{N}$[3], they are a barrier for the analytic continuation of the log-partition function, as there exist *complex zeros* of the partition function in any neighbourhood of $\{\sigma_j\}_{j=1}^d$.

The aim of this chapter is to give a detailed answer to Question 2.2.1 by

1. Identifying the distribution (=fluctuations) of the suitably rescaled $\mathcal{Z}_n(\beta)$, $\beta \in \mathbb{C}$ as $n \to \infty$.
2. Identifying the distribution of complex zeros of $\mathcal{Z}_n(\cdot)$, as $n \to \infty$.
3. Computing $p(\beta)$ for all $\beta \in \mathbb{C}$ and thus identifying the full phase diagram.

2.4 LIMITING LOG-PARTITION FUNCTION

In this section, we provide a formula for the limiting log-partition function of the GREM at complex temperatures. To understand this formula heuristically, consider a GREM with d levels

3 $\mathcal{Z}_n(\beta) > 0$, for any $\beta \in \mathbb{R}$.

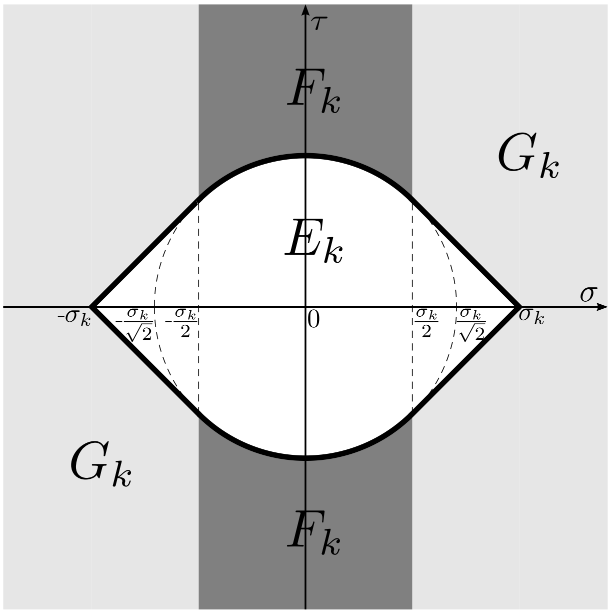


Figure 2.3: Complex-β phase diagram of the REM with the partition function $\mathcal{Z}_n^{(k)}(\beta)$; see Derrida [74] for the original heuristics and Chapter 1 for rigorous results.

as a "superposition" of d independent *random energy models*. (Note that the random field X_ε which generates the partition function of the GREM, cf. (2.9), has strong correlations.) Namely, with every level $k = 1, \ldots, d$ of the GREM we can associate a REM whose partition function is given by

$$\mathcal{Z}_n^{(k)}(\beta) = \sum_{j=1}^{N_{n,k}} \mathrm{e}^{\beta\sqrt{na_k}\eta_j^{(k)}}, \quad 1 \le k \le d, \tag{2.10}$$

where $\eta_1^{(k)}, \eta_2^{(k)}, \ldots, \eta_{N_{n,k}}^{(k)}$ are independent real standard normal random variables. The complex plane phase diagram of the REM was described by Derrida [74]; see also [123] for rigorous proofs and more refined results. In the REM (2.10), there are three phases, see Figure 2.3, which we will denote by

(a) E_k **[Expectation Dominated Phase]**,

(b) F_k **[Fluctuations Dominated Phase]**,

(c) G_k **["Glassy Phase" (= Extreme Values Dominated Phase)]**.

Analytically, these phases are specified as

$$G_k := \{\beta \in \mathbb{C} \colon 2|\sigma| > \sigma_k,\ |\sigma| + |\tau| > \sigma_k\}, \tag{2.11}$$
$$F_k := \{\beta \in \mathbb{C} \colon 2|\sigma| < \sigma_k,\ 2(\sigma^2 + \tau^2) > \sigma_k^2\}, \tag{2.12}$$
$$E_k := \mathbb{C} \setminus \overline{G_k \cup F_k}, \tag{2.13}$$

where $\bar{A}$ denotes the closure of the set A (in the Euclidean topology). The phases G_k and E_k intersect the real axis, while the phase F_k is special for the complex β case. By definition, the sets G_k, F_k, E_k are open.

Derrida [74], see Chapter 1 for a rigorous proof, computed the limiting log-partition function of the REM at complex β. Namely, for the limiting *log-partition function* of the REM corresponding to the k-th level of the GREM,

$$p_k(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n^{(k)}(\beta)| \tag{2.14}$$

(where the limit is in probability), Derrida's formula takes the form

$$p_k(\beta) = \begin{cases} |\sigma|\sqrt{2a_k \log \alpha_k}, & \text{if } \beta \in \bar{G}_k, \\ \frac{1}{2}\log \alpha_k + a_k\sigma^2, & \text{if } \beta \in \bar{F}_k, \\ \log \alpha_k + \frac{1}{2}a_k(\sigma^2 - \tau^2), & \text{if } \beta \in \bar{E}_k. \end{cases} \tag{2.15}$$

It is easy to check that the function p_k is continuous and strictly positive. In particular, p_k does not have jumps on the boundaries of the phases. In Section 2.5, we provide heuristics behind (2.15).

Our first result states that the limiting log-partition function of the GREM can be computed as the sum of the limiting log-partition functions of the REM's corresponding to the d levels of the GREM:

Theorem 2.4.1 (Log-partition function, Free Energy)**.** *For every $\beta \in \mathbb{C}$, the following limit exists in probability and in L^q, for all $q \geq 1$:*

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n(\beta)| = \sum_{k=1}^{d} p_k(\beta), \tag{2.16}$$

where $p_k(\beta)$ is the contribution of the k-th level, cf., (2.15)*.*

Remark 2.4.1. *Restricting* (2.16) *and* (2.15) *to the real temperature case $\beta \geq 0$, we recover* (2.7) *which was rigorously proved by Capocaccia et al. [50] and generalized by Bovier and Kurkova [40, 45].*

2.5 HEURISTICS

There are three natural guesses on the asymptotic behavior of $\mathcal{Z}_n^{(k)}(\beta)$:

(a) **[Expectation Dominated Phase]** $\mathcal{Z}_n^{(k)}(\beta)$ behaves approximately as its *expectation*; see Figure 2.4, left. This guess turns out to be correct in phase E_k.

However, it can happen that the fluctuations of $\mathcal{Z}_n^{(k)}(\beta)$ around its expectation are of larger order than the expectation. In this case, we end up in the following regime:

(b) **[Fluctuations Dominated Phase]** $\mathcal{Z}_n^{(k)}(\beta)$ behaves approximately as its *standard deviation*; see Figure 2.4, right. This guess turns out to be correct in phase F_k.

Still, it can happen that due to the presence of heavy tails neither the expectation nor the standard deviation are adequate to estimate the true magnitude of the partition function. In this case, one can make the following guess:

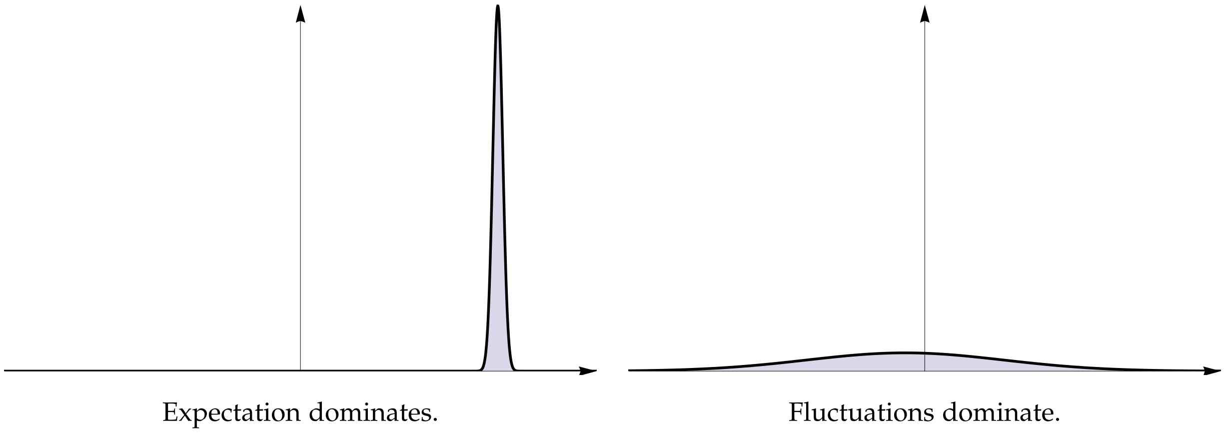


Figure 2.4: Caricatures of the probability density of $\mathcal{Z}_n^{(k)}(\beta)$ in the regimes with light tails.

(c) **[Extremes Dominated Phase]** $\mathcal{Z}_n^{(k)}(\beta)$ behaves approximately as the *maximal summand* in (2.10). This guess turns out to be correct in phase G_k.

Summarizing, we arrive at the following three guesses for the limiting log-partition function $p_k(\beta) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{Z}_n^{(k)}(\beta)|$:

$$\textbf{[Expectation]} \quad p_k(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \left| \mathbb{E}\mathcal{Z}_n^{(k)}(\beta) \right| = \log \alpha_k + \frac{1}{2} a_k(\sigma^2 - \tau^2), \tag{2.17}$$

$$\textbf{[Fluctuations]} \quad p_k(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \sqrt{\operatorname{Var} \mathcal{Z}_n^{(k)}(\beta)} = \frac{1}{2} \log \alpha_k + a_k \sigma^2, \tag{2.18}$$

$$\textbf{[Extremes]} \quad p_k(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \max_{j\in\{1,\dots,N_{n,k}\}} \left| e^{\beta\sqrt{n a_k}\eta_j^{(k)}} \right| = |\sigma| \sqrt{2 a_k \log \alpha_k}. \tag{2.19}$$

It turns out that these formulae indeed give the correct value of $p_k(\beta)$ in phases E_k, F_k, G_k, respectively, cf., (2.15).

2.6 PHASE DIAGRAM

We can now describe the *phase diagram* of the GREM in the complex β plane; see Figure 2.5. It is obtained as a superposition of the phase diagrams of the corresponding REM's. Take some $\beta \in \mathbb{C}$. For every $k = 1, \dots, d$, we can determine the phase (G_k, F_k, or E_k) to which β belongs and write the result in form of a sequence of length d over the alphabet $\{G, F, E\}$. However, it is easy to see that only phases of the following form are possible:

$$G^{d_1} F^{d_2} E^{d_3} = \underbrace{G \dots G}_{d_1} \underbrace{F \dots F}_{d_2} \underbrace{E \dots E}_{d_3}, \tag{2.20}$$

where $d_1, d_2, d_3 \in \{0, \dots, d\}$ are such that $d_1 + d_2 + d_3 = d$. In other words, we have an ordering of the level phases which can symbolically be expressed as

$$G \succ F \succ E. \tag{2.21}$$

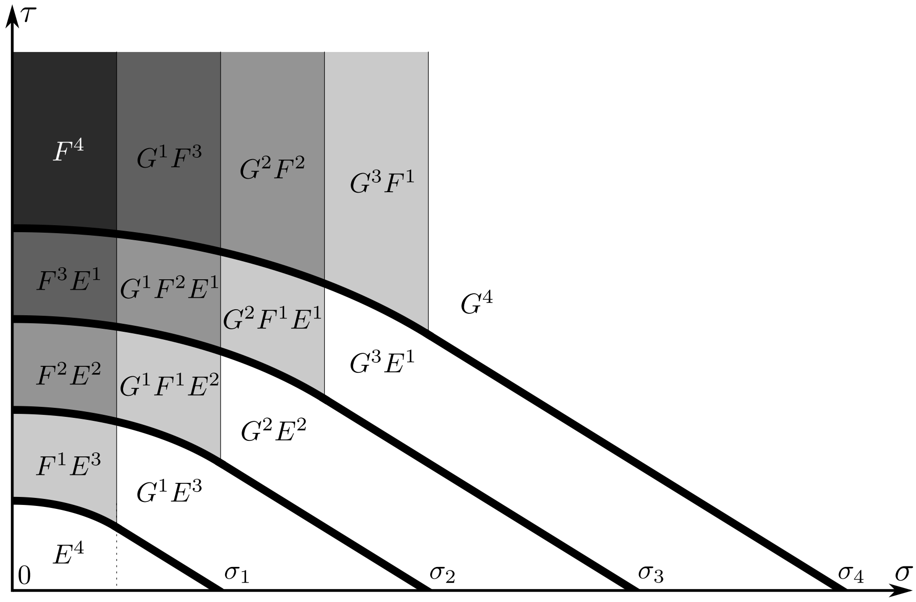


Figure 2.5: Phase diagram of a GREM with $d = 4$ levels in the complex β plane. Only the quarter-plane $\sigma \geq 0$, $\tau \geq 0$ is shown. Darker regions have larger density of partition function zeros.

For example, it is not possible that a level in E-phase is followed by a level in F- or in G-phase. This stems from the fact that if $\beta \in E_k$ for some k, then $\beta \notin F_l$ and $\beta \notin G_l$ for $l \geq k$. This ordering of phases agrees with the observation of Saakian [175]. The phases of the GREM are therefore given by

$$G^{d_1}F^{d_2}E^{d_3} = (G_1 \cap \ldots \cap G_{d_1}) \cap (F_{d_1+1} \cap \ldots \cap F_{d_1+d_2}) \cap (E_{d_1+d_2+1} \cap \ldots \cap E_d),$$

where $d_1, d_2, d_3 \in \{0, \ldots, d\}$ are such that $d_1 + d_2 + d_3 = d$. If $\beta \in G^{d_1}F^{d_2}E^{d_3}$, then we say that the levels $1, \ldots, d_1$ are in the G-phase, the levels $d_1 + 1, \ldots, d_1 + d_2$ are in the F-phase, and the levels $d_1 + d_2 + 1, \ldots, d$ are in the E-phase.

Note that each $G^{d_1}F^{d_2}E^{d_3}$ is an open subset of the complex plane and the union of the closures of these sets is the entire complex plane. Hence, phases (2.20) provide the complete phase diagram of the GREM. The total number of phases is $\frac{1}{2}(d+1)(d+2)$. Only $d+1$ of these phases, namely those of the form $G^{d_1}E^{d_3}$, intersect the real axis. Therefore, on the real inverse temperature axis, there are only $d+1$ phases "visible" which certainly agrees with what is known about the GREM at real temperatures.

2.7 MACROSCOPIC LIMITING DISTRIBUTION OF COMPLEX ZEROS

Using Theorem 2.4.1, it is possible to obtain the limiting distribution of complex zeros of the GREM partition function $\mathcal{Z}_n(\beta)$.

For the partition function of the REM $\mathcal{Z}_n^{(k)}(\beta)$, the limiting distribution of zeros has been heuristically computed by Derrida [74]; see Chapter 1 for rigorous results. The main idea is to use the Poincaré-Lelong formula (see, e.g., [111, p. 2.4.1]). It states that the measure counting the complex zeros of any analytic function f (which is not everywhere 0) can be represented as

$$\mathbf{Zeros}\{f(\beta)\colon \beta \in \mathbb{C}\} = \frac{1}{2\pi}\Delta \log |f(\beta)|. \tag{2.22}$$

Here, $\Delta = \frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}$ is the Laplace operator in the complex β-plane. The Laplace operator should be understood in the sense of generalized functions (= distributions). Applying this formula to $\mathcal{Z}_n^{(k)}(\beta)$, dividing by n, interchanging the large n limit and the Laplacian (which should be justified), and using (2.14), one can show, see [123], that weakly on $\mathbb{M}(\mathbb{C})$[4],

$$\frac{1}{n}\mathbf{Zeros}\{\mathcal{Z}_n^{(k)}(\beta)\colon \beta \in \mathbb{C}\} \xrightarrow[N\to\infty]{w} \frac{1}{2\pi}\Delta p_k. \tag{2.23}$$

The distributional Laplacian of p_k is a measure Ξ_k on $\mathbb{C}$ given by

$$\Xi_k := \Delta p_k = \Xi_k^F + \Xi_k^{EF} + \Xi_k^{EG}, \tag{2.24}$$

where Ξ_k^F, Ξ_k^{EF}, Ξ_k^{EG} are measures on the complex plane defined as follows:

(a) Ξ_k^F is $2a_k$ times the two-dimensional Lebesgue measure restricted to F_k.

(b) Ξ_k^{EF} is $\sqrt{a_k \log \alpha_k}$ times the one-dimensional length measure on the boundary between E_k and F_k (which consists of two circular arcs).

(c) Ξ_k^{EG} is a measure having the density $\sqrt{2}a_k|\tau|$ with respect to the one-dimensional length measure restricted to the boundary between E_k and G_k (which consists of four line segments).

Thus, the zeros of $\mathcal{Z}_n^{(k)}(\beta)$ fill the *two-dimensional* region F_k asymptotically uniformly with density $2a_k n$, but some zeros concentrate around the boundary of E_k with *one-dimensional* density asymptotically proportional to n. The term Ξ_k^F is just the pointwise Laplacian of p_k, whereas the terms Ξ_k^{EF} and Ξ_k^{EG} appear because the normal derivative of the function p_k has a jump discontinuity on the boundary of the phase E_k. On the boundary between F_k and G_k, the normal derivative of p_k is continuous, hence this boundary makes no one-dimensional contribution to Ξ.

We now proceed to the complex zeros of $\mathcal{Z}_n(\beta)$. In view of Theorem 2.4.1, it is not surprising that the limiting distribution of zeros of $\mathcal{Z}_n(\beta)$ can be obtained as a *superposition* of the limiting zeros distributions of the corresponding REM's.

Theorem 2.7.1 (Global Distribution of Zeros). *The following convergence of random measures holds weakly on* $\mathbb{M}(\mathbb{C})$*:*

$$\frac{1}{n}\mathbf{Zeros}\{\mathcal{Z}_n(\beta)\colon \beta \in \mathbb{C}\} \xrightarrow[N\to\infty]{w} \frac{1}{2\pi}\Xi, \tag{2.25}$$

where $\Xi = \Delta p = \sum_{k=1}^d \Xi_k$.

4 $\mathbb{M}(\mathbb{C})$ denotes the space of locally finite measures on $\mathbb{C}$. We endow $\mathbb{M}(\mathbb{C})$ with the vague topology.

2.8 FUNCTIONAL LIMIT THEOREMS AND LOCAL STRUCTURE OF ZEROS

One may ask whether the partition function $\mathcal{Z}_n(\beta)$ converges, weakly on a suitable function space, to some limiting random analytic function. In this section, we state functional central limit theorems (FCLT) of this type. In order to obtain a non-trivial functional convergence, we have to rescale the partition function appropriately.

TRUNCATED EXPECTATION DOMINATED PHASE. We focus on the *truncated expectation dominated phase* defined by

$$\hat{E} := \left\{ \beta = \sigma + i\tau \in \mathbb{C} \colon |\sigma| < \frac{\sigma_1}{2}, |\beta| < \frac{\sigma_1}{\sqrt{2}} \right\}, \tag{2.26}$$

Theorem 2.8.1 (Functional LLN on $\hat{E}$)**.** *The following convergence of random analytic functions holds weakly on $\mathcal{H}(\hat{E})$:*

$$\frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\mathcal{Z}_n(\beta)} \xrightarrow[N\to\infty]{w} 1. \tag{2.27}$$

We immediately obtain

Corollary 2.8.1 (Absence of Zeros in $\hat{E}$)**.** *The following weak convergence of point processes on $\mathcal{N}(\hat{E})$ holds:*

$$\mathbf{Zeros}\{\mathcal{Z}_n(\beta) \colon \beta \in \hat{E}\} \xrightarrow[N\to\infty]{w} \varnothing. \tag{2.28}$$

Here, $\varnothing$ denotes the empty point process on $\hat{E}$.

In the next theorem, we will obtain more refined results than in Theorem 2.8.1 by a "better" choice of normalization. Essentially, we describe the limiting fluctuations of $\mathcal{Z}_n(\beta)$ around its expected value $\mathbb{E}\mathcal{Z}_n(\beta)$. The limiting fluctuations are given by the *planar Gaussian analytic function* $\mathbb{X}$; see [111, 182]. It is a random analytic function $\{\mathbb{X}(t) \colon t \in \mathbb{C}\}$ given by

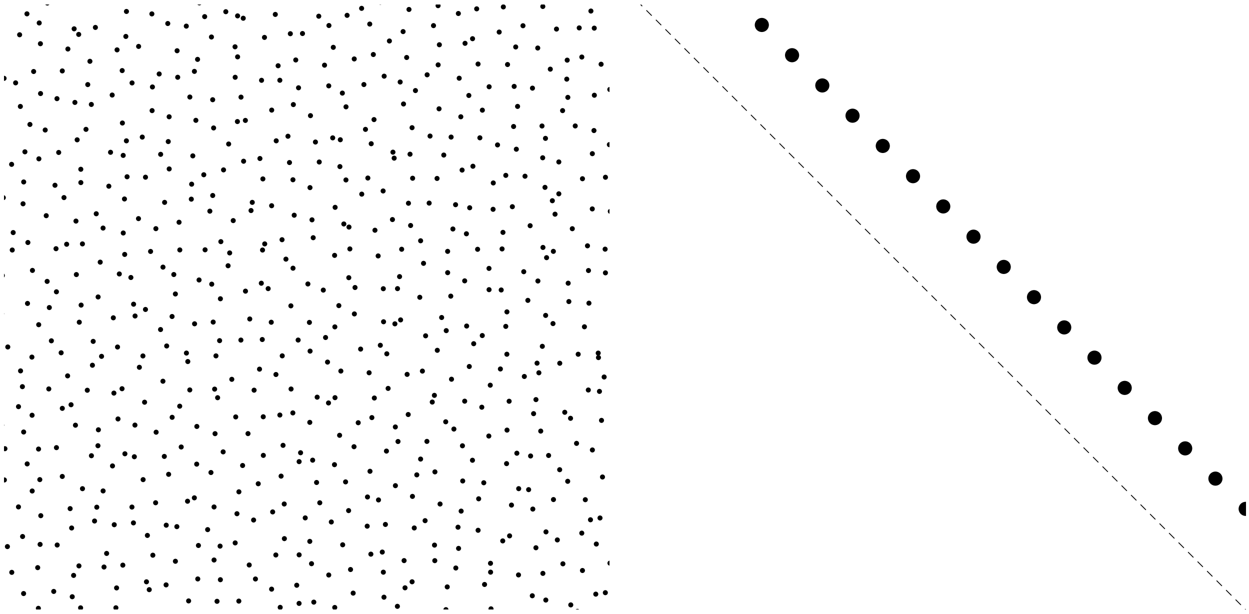
$$\mathbb{X}(t) = \mathrm{e}^{-\frac{t^2}{2}} \sum_{k=0}^{\infty} N_k \frac{t^k}{\sqrt{k!}}, \tag{2.29}$$

where $N_1, N_2, \ldots \sim N_{\mathbb{C}}(0,1)$ are independent complex standard Gaussian random variables. The finite-dimensional distributions of $\mathbb{X}$ are multivariate complex Gaussian distributions and the second-order structure of $\mathbb{X}$ is given by

$$\mathbb{E}\mathbb{X}(t) = 0, \quad \mathbb{E}[\mathbb{X}(t_1)\mathbb{X}(t_2)] = 0, \quad \mathbb{E}[\mathbb{X}(t_1)\overline{\mathbb{X}(t_2)}] = \mathrm{e}^{-\frac{1}{2}(t_1-\bar{t}_2)^2}, \quad t_1, t_2 \in \mathbb{C}. \tag{2.30}$$

The restriction of $\mathbb{X}$ to $\mathbb{R}$ is a stationary complex Gaussian process. The factor $\mathrm{e}^{-t^2/2}$ in (2.29) is chosen to simplify the statements of our results and is usually not used in the literature. The set of complex zeros of $\mathbb{X}$ is a remarkable *stationary* point process; see Figure 2.6, left. The intensity of this point process is π^{-1}, that is for every Borel set $B \subset \mathbb{C}$ we have

$$\mathbb{E}\left[\sum_{z\in B} \mathbb{1}_{\mathbb{X}(z)=0}\right] = \frac{1}{\pi}\operatorname{Leb}(B). \tag{2.31}$$



Zeros of the plane Gaussian Analytic Function. Note the stationarity of the distribution of zeros.

"Curve of zeros" seen locally. The dotted line is the boundary between the phases. Note the "periodicity" of zeros.

Figure 2.6: Point processes of zeros.

For more information on the zeros of $\mathbb{X}$, we refer to [111, 182].

We are ready to state the functional central limit theorem in the domain $\hat{E}$. Define normalizing functions $\hat{g}_n(\beta_*; t)$, with $t \in \mathbb{C}$, by

$$\hat{g}_n(\beta_*; t) := \left(\frac{1}{2}\log N_{n,1} + a_1(\sqrt{n}\sigma_* + t)^2\right) + \sum_{l=2}^{d}\left(\log N_{n,l} + \frac{1}{2}a_l(\sqrt{n}\beta_* + t)^2\right). \tag{2.32}$$

Theorem 2.8.2 (FCLT on $\hat{E}$). *Fix $\beta_* = \sigma_* + i\tau_* \in \hat{E}$ with $\tau_* \neq 0$. Then, the following convergence of random analytic functions holds weakly on $\mathcal{H}(\mathbb{C})$:*

$$\left\{ e^{-\hat{g}_n(\beta_*;t)} \left(\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right) - \mathbb{E}\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right)\right) : t \in \mathbb{C} \right\} \xrightarrow[N\to\infty]{w} \{\mathbb{X}(\sqrt{a_1}t) \colon t \in \mathbb{C}\}, \tag{2.33}$$

where $\{\mathbb{X}(t) \colon t \in \mathbb{C}\}$ is the plane Gaussian analytic function (2.29).

Remark 2.8.1. *In the case of real $\beta_* \in (-\frac{\sigma_1}{2}, \frac{\sigma_1}{2})$, an analogue of Theorem 2.8.2 is valid, but the limiting process is $\{\mathbb{X}_{\mathbb{R}}(\sqrt{a_1}t) \colon t \in \mathbb{C}\}$, where the random analytic function $\mathbb{X}_{\mathbb{R}}$ is the real analogue of $\mathbb{X}$ defined by*

$$\mathbb{X}_{\mathbb{R}}(t) = e^{-\frac{t^2}{2}} \sum_{k=0}^{\infty} N_k \frac{t^k}{\sqrt{k!}}, \tag{2.34}$$

where $N_1, N_2, \ldots \sim N_{\mathbb{R}}(0,1)$ are independent real standard Gaussian random variables.

2.9 POISSON CASCADE ZETA FUNCTION

The fluctuations of $\mathcal{Z}_n(\beta)$ in phases of the form $G^{d_1}F^{d_2}E^{d_3}$ with $d_1 > 0$ (at least one glassy level) will be described using a random zeta function associated to the Poisson cascades. In this section, we define this function and state results on its meromorphic continuation.

Let $P_1, P_2, \dots$ be the points of a unit intensity Poisson point process on $(0, \infty)$. The points are always arranged in an increasing order: $0 < P_1 < P_2 < \dots$. The *Poisson process zeta function* is defined by

$$\zeta_P(z) = \sum_{k=1}^{\infty} P_k^{-z}, \quad \operatorname{Re} z > 1. \tag{2.35}$$

With probability 1, the above series converges absolutely and uniformly on compact subsets of the half-plane $\{\operatorname{Re} z > 1\}$ since $\lim_{k\to\infty} P_k/k = 1$ a.s. by the law of large numbers. So, ζ_P is an analytic function on the half-plane $\{\operatorname{Re} z > 1\}$, a.s. Moreover, with probability 1, the function ζ_P admits a meromorphic continuation to the half-plane $\{\operatorname{Re} z > 1/2\}$. Namely, by [123, Theorem 2.6], with probability 1, we have

$$\sum_{P_k \le T} P_k^{-z} - \int_1^T t^{-z}\mathrm{d}t \underset{T\to\infty}{\longrightarrow} \zeta_P(z) - \frac{1}{z-1} \quad \text{on } \mathcal{H}\left(\{\operatorname{Re} z > 1/2\}\right), \tag{2.36}$$

where $\mathcal{H}(D)$ denotes the space of analytic functions on a domain D endowed with the topology of locally uniform convergence.

We will need a multivariate generalization of the Poisson process zeta function which will be called the *Poisson cascade zeta function*. First, we need to define the Poisson cascade point processes; see Figure 2.7. These and related point processes appeared for example in [40], [174]. Fix dimension $d \in \mathbb{N}$. Start with a unit intensity Poisson point process $\sum_{i=1}^{\infty} \delta(P_i)$ on $(0, \infty)$. Then, for every $m = 1, \dots, d-1$ and every $\varepsilon_1, \dots, \varepsilon_m \in \mathbb{N}$ let $\sum_{i=1}^{\infty} \delta(P_{\varepsilon_1 \dots \varepsilon_m i})$ be a unit intensity Poisson point process on $(0, \infty)$. Assume that all point processes introduced above are independent. Consider the following point process Π on $(0, \infty)^d$,

$$\Pi = \sum_{\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \mathbb{N}^d} \delta(P_{\varepsilon_1}, P_{\varepsilon_1\varepsilon_2}, \dots, P_{\varepsilon_1\dots\varepsilon_d}). \tag{2.37}$$

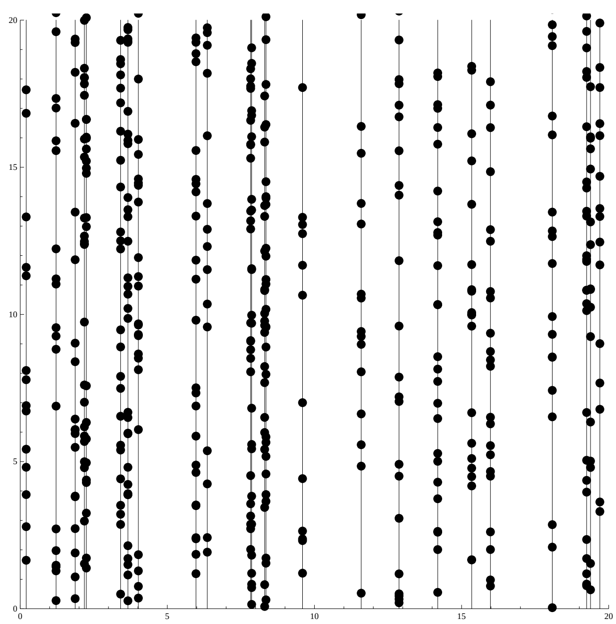
Of course, Π is not a Poisson process (unless $d = 1$) since Π contains infinitely many collinear points with probability 1. The next lemma states that Π has the same first order intensity as the homogeneous Poisson process on $(0, \infty)^d$. It can easily be proven by induction over d.

Lemma 2.9.1. *Let φ be an integrable or non-negative function on $(0, \infty)^d$. Then,*

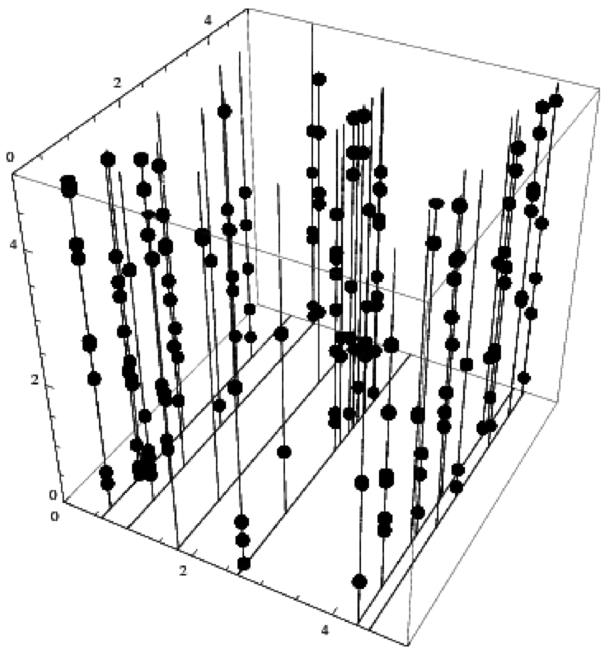
$$\mathbb{E}\left[\sum_{x \in \Pi} \varphi(x)\right] = \int_{(0,\infty)^d} \varphi(x)\mathrm{d}x. \tag{2.38}$$

The random zeta function ζ_P associated to the Poisson cascade point process Π is a stochastic process defined by the series

$$\zeta_P(z_1, \dots, z_d) = \sum_{\varepsilon \in \mathbb{N}^d} P_{\varepsilon_1}^{-z_1} P_{\varepsilon_1\varepsilon_2}^{-z_2} \dots P_{\varepsilon_1\dots\varepsilon_d}^{-z_d}. \tag{2.39}$$



$d = 2$ levels.



$d = 3$ levels.

Figure 2.7: Poisson cascade point process.

Theorem 2.9.1 (Domain of ζ_P)**.** *With probability* 1*, the series* (2.39) *converges absolutely and uniformly on any compact subset of the domain*

$$\mathcal{D} := \{(z_1, \dots, z_d) \in \mathbb{C}^d \colon \operatorname{Re} z_1 > \dots > \operatorname{Re} z_d > 1\}. \tag{2.40}$$

In particular, the function ζ_P is analytic on $\mathcal{D}$ with probability 1.

Theorem 2.9.1 would be sufficient to treat the GREM at real inverse temperature β, as in [40]. However, for complex β, we need a meromorphic continuation of ζ_P to a larger domain.

Theorem 2.9.2 (Meromorphic Continuation of ζ_P)**.** *With probability* 1*, the function $\zeta_P(z_1, \dots, z_d)$ defined originally on $\mathcal{D}$ admits a meromorphic continuation to the domain*

$$\frac{1}{2}\mathcal{D} = \{(z_1, \dots, z_d) \in \mathbb{C}^d \colon \operatorname{Re} z_1 > \dots > \operatorname{Re} z_d > 1/2\}. \tag{2.41}$$

Moreover, the function $(z_d - 1)\zeta_P(z_1, \dots, z_d)$ is analytic on $\frac{1}{2}\mathcal{D}$ with probability 1.

We conjecture that with probability 1 there is no meromorphic continuation beyond $\frac{1}{2}\mathcal{D}$. In the sequel, we use the notation $z = (z_1, \dots, z_d) \in \mathbb{C}^d$.

Remark 2.9.1. *The value of $(z_d - 1)\zeta_P(z)$ in the case $z_d = 1$ is understood by continuity. In the case $d = 1$, this value is equal to* 1 *a.s., whereas, for $d \geq 2$, it is a non-degenerate random variable. (The non-degeneracy follows from the fact that a degenerate random variable cannot satisfy* (2.42)*, see below, with* $\operatorname{Re} z_1 > z_d = 1$*).*

Proposition 2.9.1 (Stability of ζ_P)**.** *Consider $m \in \mathbb{N}$ independent copies of the random analytic function $\{(z_d - 1)\zeta_P(z) \colon z \in \frac{1}{2}\mathcal{D}\}$ denoted by $\{(z_d - 1)\zeta_P^{(j)}(z) \colon z \in \frac{1}{2}\mathcal{D}\}$, $1 \leq j \leq m$. Then, the following distributional equality on $\mathcal{H}(\frac{1}{2}\mathcal{D})$ holds:*

$$\left\{ \sum_{j=1}^{m} (z_d - 1)\zeta_P^{(j)}(z) \colon z \in \frac{1}{2}\mathcal{D} \right\} \stackrel{d}{=} \left\{ m^{z_1}(z_d - 1)\zeta_P(z) \colon z \in \frac{1}{2}\mathcal{D} \right\}. \tag{2.42}$$

From Proposition 2.9.1, we can draw several conclusions about the finite-dimensional distributions of ζ_P. If $z \in \frac{1}{2}\mathcal{D} \cap \mathbb{R}^d$, then the distribution of the real-valued random variable $(z_d - 1)\zeta_P(z)$ is *stable* with exponent $1/z_1$; see [178, Chapter 1]. In fact, it is even *strictly stable* meaning that no additive constant is needed in (2.42). If $z \in \frac{1}{2}\mathcal{D}$ is such that $z_1 \in \mathbb{R}$ (but $z_2, \dots, z_d$ are not necessarily real), then the term m^{z_1} is real and hence, $(z_d - 1)\zeta_P(z)$ (which is considered as a random vector with values in $\mathbb{C} \equiv \mathbb{R}^2$) has a two-dimensional stable distribution (which need not be isotropic); see [178, Chapter 2]. In general, for $z \in \frac{1}{2}\mathcal{D}$ without any additional assumptions on the components, the distribution of the random variable $(z_d - 1)\zeta_P(z)$ (again considered as a random vector with values in $\mathbb{C} \equiv \mathbb{R}^2$) is *strictly complex stable* in the sense of Hudson and Veeh [112]. A random variable with values in $\mathbb{C}$ is called strictly complex stable, see [112], if for every $m \in \mathbb{N}$ the sum of m independent copies of this random variable, after dividing it by an appropriate complex number, has the same law as the original random variable. More generally, all finite-dimensional distributions of the stochastic process $\{(z_d - 1)\zeta_P(z) \colon z \in \frac{1}{2}\mathcal{D}\}$ are *strictly operator stable* (and hence, infinitely divisible). Recall that a random vector with values in $\mathbb{R}^k$ is called strictly operator stable, if for every $m \in \mathbb{N}$ the sum of m copies of this random vector, after applying to it an appropriate linear transformation of $\mathbb{R}^k$, has the same law as the original random vector; see [147, Definition 3.3.24]. The same conclusions apply to the random variable $\zeta_P(z)$ and the stochastic process $\{\zeta_P(z) \colon z \in \frac{1}{2}\mathcal{D}\}$ if we additionally assume that $z_d \neq 1$.

Proposition 2.9.2 (Moments of ζ_P). *Let $0 < p < 2$ and $z \in \frac{1}{2}\mathcal{D}$.*

(1) *If $\operatorname{Re} z_1 < \frac{1}{p}$, then $\mathbb{E}|(z_d - 1)\zeta_P(z)|^p < \infty$.*

(2) *If $\operatorname{Re} z_1 > \frac{1}{p}$, then $\mathbb{E}|(z_d - 1)\zeta_P(z)|^p = \infty$ (unless $d = 1$ and $z = 1$).*

2.10 FLUCTUATIONS OF THE PARTITION FUNCTION

In this section, we aim to describe the limiting fluctuations of $\mathcal{Z}_n(\beta)$. First, we need to introduce several normalizing sequences. For each $1 \le k \le d$, let $\{u_{n,k}\}_{n\in\mathbb{N}}$ be a real sequence satisfying

$$N_{n,k} \underset{n\to\infty}{\sim} \sqrt{2\pi} u_{n,k} e^{\frac{1}{2}u_{n,k}^2}. \tag{2.43}$$

Equivalently, we can choose

$$u_{n,k} \underset{n\to\infty}{=} \sqrt{2\log N_{n,k}} - \frac{\log(4\pi \log N_{n,k}) + o(1)}{2\sqrt{2\log N_{n,k}}} \underset{n\to\infty}{\sim} \sqrt{2n\log\alpha_k} = \sigma_k\sqrt{na_k}. \tag{2.44}$$

It is a well-known fact from extreme value theory, see [136, Theorem 1.5.3], that if $\eta_1, \eta_2, \ldots$ are independent real standard Gaussian random variables, then

$$u_{n,k}\left(\max_{i=1,\ldots,N_{n,k}} \eta_i - u_{n,k}\right) \xrightarrow[N\to\infty]{w} e^{-e^{-x}}. \tag{2.45}$$

Let $\beta \in \mathbb{C}$ be located inside (but not on the boundary) of some phase $G^{d_1}F^{d_2}E^{d_3}$ and let $\sigma \ge 0$. For $1 \le k \le d$, to scale the k-th level of the GREM, we define a sequence of functions $c_{n,k}(\beta)$ by

$$c_{n,k}(\beta) := \begin{cases} \beta\sqrt{na_k}\, u_{n,k}, & \text{if } \beta \in G_k, \\ \frac{1}{2}\log N_{n,k} + a_k\sigma^2 n, & \text{if } \beta \in F_k, \\ \log N_{n,k} + \frac{1}{2}a_k\beta^2 n, & \text{if } \beta \in E_k. \end{cases} \tag{2.46}$$

Then, define a normalizing function $c_n(\beta)$ by

$$c_n(\beta) := c_{n,1}(\beta) + \ldots + c_{n,d}(\beta). \tag{2.47}$$

The next theorem describes the fluctuations of $\mathcal{Z}_n(\beta)$ inside the phases. The boundary cases will be studied in detail later.

Theorem 2.10.1 (Fluctuations of the Partition Function). *Let $\beta \in G^{d_1}F^{d_2}E^{d_3}$ and let $\sigma \ge 0$. Then,*

$$\frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} \xrightarrow[N\to\infty]{w} \begin{cases} 1, & \text{if } d_1 = 0 \text{ and } d_2 = 0, \\ N_{\mathbb{C}}(0,1), & \text{if } d_1 = 0 \text{ and } d_2 > 0, \\ \zeta_P(\frac{\beta}{\sigma_1}, \ldots, \frac{\beta}{\sigma_{d_1}}), & \text{if } d_1 > 0 \text{ and } d_2 = 0, \\ cS_{\sigma_1/\sigma}, & \text{if } d_1 > 0 \text{ and } d_2 > 0. \end{cases} \tag{2.48}$$

Here, ζ_P is the Poisson cascade zeta function; S_α is the rotationally invariant, complex standard α-stable random variable with characteristic function $\mathbb{E}\left[e^{i\operatorname{Re}(S_\alpha\bar{z})}\right] = e^{-|z|^\alpha}$, $z \in \mathbb{C}$, where $\alpha \in (0,2)$; and c is a constant.

Proof. We will establish stronger results below. The case $d_1 = 0$, $d_2 = 0$ follows from Theorem 2.11.1. The case $d_1 > 0$, $d_2 = 0$ follows from Theorem 2.11.4 below. The case $d_1 > 0$, $d_2 > 0$ follows from Theorem 2.11.5 (with $t = 0$) below. □

Remark 2.10.1. *In the case $d_1 = d_2 = 0$ (i.e., $\beta \in E^d = E_1$), the limit in (2.48) is degenerate. Under a more refined normalization, it can be that $\mathcal{Z}_n(\beta)$ has Gaussian limiting fluctuations in the cases $\beta \in E_1 \cap \{|\sigma| < \frac{\sigma_1}{2}\}$ and $|\sigma| = \frac{\sigma_1}{2}$. The fluctuations in the case $\beta \in E_1 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$ are non-Gaussian and will be identified in Theorem 2.11.3 below.*

Remark 2.10.2. *The assumption $\sigma \geq 0$ in Theorem 2.10.1 can be removed if we define*

$$c_{n,k}(\beta) := (\operatorname{sgn} \sigma) \cdot \beta \sqrt{n a_k}\, u_{n,k}, \quad \text{for } \beta \in G_k. \tag{2.49}$$

2.11 FUNCTIONAL LIMIT THEOREMS AND LOCAL STRUCTURE OF ZEROS

In this section, we state the results on the weak functional convergence of the rescaled partition function $\mathcal{Z}_n(\beta)$ to some limiting random analytic function. Since weak convergence of random analytic functions implies weak convergence of point processes of zeros any functional limit theorem of this type implies a result on the local structure of zeros of $\mathcal{Z}_n(\beta)$.

PHASE $E_1 = E^d$ The first result is a functional law of large numbers in the purely expectation dominated phase $E_1 = E^d$.

Theorem 2.11.1 (Functional LLN for the Partition Function in E_1)**.** *The following convergence of random analytic functions holds weakly on $\mathcal{H}(E_1)$:*

$$\frac{\mathcal{Z}_n(\beta)}{\mathbb{E}\,\mathcal{Z}_n(\beta)} \xrightarrow[N\to\infty]{w} 1. \tag{2.50}$$

Denote by $\mathcal{N}(D)$ the space of locally finite point measures on a locally compact metric space D. We endow $\mathcal{N}(D)$ with the topology of vague convergence.

Corollary 2.11.1 (Absence of Zeros in E_1)**.** *The following weak convergence of point processes on $\mathcal{N}(E_1)$ holds:*

$$\textbf{Zeros}\{\mathcal{Z}_n(\beta) \colon \beta \in E_1\} \xrightarrow[N\to\infty]{w} \varnothing. \tag{2.51}$$

Here, $\varnothing$ denotes the empty point process on E_1.

The above corollary follows from Theorem 2.11.1. In fact, we will prove much more: The probability that the partition function $\mathcal{Z}_n$ has a zero in any fixed compact set $K \subset E_1$ decays exponentially.

Theorem 2.11.2 (Absence of Zeros in the Expectation Phase)**.** *Let K be a compact subset of E_1. Then, there exist $C = C(K)$ and $\varepsilon = \varepsilon(K) > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}\left[\exists \beta \in K \colon \mathcal{Z}_n(\beta) = 0\right] < C e^{-\varepsilon n}. \tag{2.52}$$

It is natural to ask whether results more refined than Theorem 2.11.1 (where the limiting process is degenerate) can be obtained by a "better" choice of normalization. Functional limit theorems with Gaussian fluctuations in the strip $|\sigma| < \frac{\sigma_1}{2}$ can be obtained. Here, we concentrate on the case $|\sigma| > \frac{\sigma_1}{2}$ where the fluctuations are non-Gaussian. The next result states that in the domain $E_1 \cap \{\sigma > \frac{\sigma_1}{2}\}$ the limiting fluctuations of $\mathcal{Z}_n(\beta)$ are given by the Poisson zeta function. Recall the definition of $c_{n,k}(\beta)$ from (2.46) and define

$$\tilde{c}_n(\beta) := c_{n,2}(\beta) + \ldots + c_{n,d}(\beta). \tag{2.53}$$

Theorem 2.11.3 (Non-Gaussian Fluctuations in the Beak-shaped Part of E_1)**.** *The following convergence of random analytic functions holds weakly on $\mathcal{H}(E_1 \cap \{\sigma > \frac{\sigma_1}{2}\})$:*

$$\frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{\beta\sqrt{na_1}u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[N\to\infty]{w} \zeta_P\left(\frac{\beta}{\sigma_1}\right). \tag{2.54}$$

Remark 2.11.1. *By symmetry, the following convergence of random analytic functions holds weakly on $\mathcal{H}(E_1 \cap \{\sigma < -\frac{\sigma_1}{2}\})$:*

$$\frac{\mathcal{Z}_n(\beta) - \mathbb{E}\mathcal{Z}_n(\beta)}{e^{-\beta\sqrt{na_1}u_{n,1} + \tilde{c}_n(\beta)}} \xrightarrow[N\to\infty]{w} \zeta_P^-\left(-\frac{\beta}{\sigma_1}\right), \tag{2.55}$$

where ζ_P^- is a copy of ζ_P. In fact, one can even show that the functional limit theorem holds on the union of both domains, namely $E_1 \cap \{|\sigma| > \frac{\sigma_1}{2}\}$, and that the limiting functions ζ_P and ζ_P^- are independent; see Remark 2.11.2 for explanation.

PHASES OF THE FORM $G^{d_1}E^{d_3}$ In the next theorem, we prove the functional convergence of the partition function $\mathcal{Z}_n(\beta)$ in the phases of the form $G^{d_1}E^{d_3}$, where $d_1, d_3 \in \{0, \ldots, d\}$ satisfy $d_1 + d_3 = d$. The limiting process is given in terms of the d_1-variate Poisson cascade zeta function ζ_P. Recall that $c_n(\beta)$ was defined in (2.47). For $1 \le l \le d$, define

$$T^l(\beta) := \left(\frac{\beta}{\sigma_1}, \ldots, \frac{\beta}{\sigma_l}\right) \in \mathbb{C}^l, \quad T^0(\beta) := \emptyset. \tag{2.56}$$

Theorem 2.11.4 (Non-Gaussian fluctuations in GE-phases)**.** *Fix some $d_1, d_3 \in \{0, \ldots, d\}$ such that $d_1 + d_3 = d$. The following convergence of random analytic functions holds weakly on $\mathcal{H}(G^{d_1}E^{d_3} \cap \{\sigma > 0\})$:*

$$\frac{\mathcal{Z}_n(\beta)}{e^{c_n(\beta)}} \xrightarrow[N\to\infty]{w} \zeta_P\left(T^{d_1}(\beta)\right). \tag{2.57}$$

In particular, for $d_1 = 0$, the limiting process is $\zeta_P(\emptyset) = 1$, and we recover Theorem 2.11.1. If the case of real β (and without functional convergence), the result of Theorem 2.11.4 is contained in Bovier and Kurkova [40, Theorem 1.7].

Remark 2.11.2. *Let $d_1 \ge 1$. By symmetry, a result similar to Theorem 2.11.4 holds in the domain $G^{d_1}E^{d_3} \cap \{\sigma < 0\}$. Namely, the following convergence of random analytic functions holds weakly on $\mathcal{H}(G^{d_1}E^{d_3} \cap \{\sigma < 0\})$:*

$$\frac{\mathcal{Z}_n(\beta)}{e^{c_n(-\beta)}} \xrightarrow[N\to\infty]{w} \zeta_P^-\left(T^{d_1}(-\beta)\right), \tag{2.58}$$

where ζ_P^- is a copy of ζ_P. One can show that (2.57) and (2.58) can be combined into a joint convergence in the phase $G^{d_1}E^{d_3}$ and that the limiting functions ζ_P and ζ_P^- are independent, for $d_1 \geq 1$. We will not provide a complete proof of the independence, but let us explain the idea. The function ζ_P in (2.57) appears as the contribution of the upper extremal order statistics among the energies on the first d_1 levels of the GREM, whereas all other levels make a deterministic contribution equal to the expectation. The function ζ_P^- in (2.58) appears as the contribution of the lower extremal order statistics on the first d_1 levels of the GREM. Since upper and lower extremal order statistics become independent in the large sample limit, the limiting functions ζ_P and ζ_P^- are independent.

Since weak convergence of random analytic functions implies weak convergence of their zero sets, Theorem 2.11.4 yields the following

Corollary 2.11.2 (Local Distribution of Zeros in GE-phases). *Under the same conditions as in Theorem 2.11.4, the following weak convergence of point processes on $\mathcal{N}(G^{d_1}E^{d_3} \cap \{\sigma > 0\})$ holds:*

$$\mathbf{Zeros}\{\mathcal{Z}_n(\beta)\} \xrightarrow[N\to\infty]{w} \mathbf{Zeros}\left\{\zeta_P\left(T^{d_1}(\beta)\right)\right\}. \tag{2.59}$$

Note that the intensity of zeros in the limiting point process is $O(1)$ as $n \to \infty$ and hence these zeros do not appear in the limit in Theorem 2.7.1. For $d_1 = 0$, the limiting point process of zeros is empty and we recover Corollary 2.11.1.

PHASES WITH AT LEAST ONE FLUCTUATION LEVEL Our next result is a functional limit theorem describing the limiting structure of the random analytic function $\mathcal{Z}_n(\beta)$ in an infinitesimal window around some $\beta_* = \sigma_* + i\tau_* \in G^{d_1}F^{d_2}E^{d_3}$, where $d_2 \geq 1$. The phases we consider have at least one fluctuation level, see the shaded regions on Figure 2.5. In the case when $d_1 = 0$ (there are no glassy levels), the limiting fluctuations are in terms of the so-called *plane Gaussian analytic function* $\mathbb{X}$ a remarkable object that appeared for example in Hough *et al.* [111] and Sodin & Tsirelson [182].

For a given $\beta_* \in G^{d_1}F^{d_2}E^{d_3}$ consider the following sequence of normalizing functions $c_n(\beta_*; t)$ that are quadratic in $t \in \mathbb{C}$ and extend $c_n(\beta_*)$, see (2.46) and (2.47), in the sense that $c_n(\beta_*; t) = c_n(\beta_*)$:

$$\begin{aligned} c_n(\beta_*; t) &:= \left(\beta_* + \frac{t}{\sqrt{n}}\right) \sum_{j=1}^{d_1} \sqrt{n a_j} u_{n,j} + \\ &+ \sum_{j=d_1+1}^{d_1+d_2} \left(\frac{1}{2}\log N_{n,j} + a_j(\sqrt{n}\sigma_* + t)^2\right) + \sum_{j=d_1+d_2+1}^{d} \left(\log N_{n,j} + \frac{1}{2}a_j(\sqrt{n}\beta_* + t)^2\right). \end{aligned} \tag{2.60}$$

Theorem 2.11.5 (Fluctuations in Phases with F-levels). *Fix some $d_1, d_2, d_3 \in \{0, \dots, d\}$ with $d_1 + d_2 + d_3 = d$ and $d_2 \geq 1$. Also, fix some $\beta_* = \sigma_* + i\tau_* \in G^{d_1}F^{d_2}E^{d_3}$ such that $\sigma_* \geq 0$. Then, the following convergence of random analytic functions holds weakly on $\mathcal{H}(\mathbb{C})$:*

$$\left\{e^{-c_n(\beta_*;t)}\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right) : t \in \mathbb{C}\right\} \xrightarrow[N\to\infty]{w} \left\{\sqrt{W}\,\mathbb{X}(\kappa t) : t \in \mathbb{C}\right\}, \tag{2.61}$$

where

(1) $W = \zeta_P(2T^{d_1}(\sigma_*))$ *and* ζ_P *is the Poisson cascade zeta function with* d_1 *variables;*

(2) $\{\mathbb{X}(t) : t \in \mathbb{C}\}$ *is the plane Gaussian analytic function* (2.29)*;*

(3) $\kappa^2 = \sum_{k=1}^{d_2} a_{d_1+k}$ *is the total variance of the GREM levels which are in the fluctuation phase;*

(4) *the processes* ζ_P *and* $\mathbb{X}$ *are independent.*

The term $\mathbb{X}(\kappa t)$ represents the contribution of the F-levels, the term $\sqrt{W}$ is the contribution of the G-levels, whereas the contribution of the E-levels is deterministic and is contained in the normalization sequence $c_n(\beta_*; t)$. If $d_1 = 0$ (i.e., there are no levels in the glassy phase), then the limit is the Gaussian analytic function $\mathbb{X}(\kappa t)$ since we have $\zeta_P(\varnothing) = 1$. In the case $d_1 \neq 0$, the limiting process is a Gaussian process rescaled by the square root of an independent real $\frac{\sigma_1}{2\sigma_*}$-stable random variable $W = \zeta_P(2T^{d_1}(\sigma_*))$ with skewness parameter $+1$. Such a process is itself complex $\frac{\sigma_1}{\sigma_*}$-stable with complex isotropic margins. Processes of this type are called *subgaussian*; see Samorodnitsky & Taqqu [178].

Corollary 2.11.3 (Local Distribution of Zeros in Phases with F-levels)**.** *Under the same conditions as in Theorem* 2.11.5*, the following convergence of the point processes of zeros holds weakly on* $\mathcal{N}(\mathbb{C})$*:*

$$\textbf{Zeros}\left\{\mathcal{Z}_n\left(\beta_* + \frac{t}{\sqrt{n}}\right) : t \in \mathbb{C}\right\} \xrightarrow[N\to\infty]{w} \textbf{Zeros}\left\{\mathbb{X}(\kappa t) : t \in \mathbb{C}\right\}. \tag{2.62}$$

Given that the intensity of complex zeros of $\{\mathbb{X}(\kappa t) : t \in \mathbb{C}\}$ is $\pi^{-1}\kappa^2$, Corollary 2.11.3 suggests that in phase $G^{d_1}F^{d_2}E^{d_3}$ with $d_2 \geq 1$ the density of complex zeros of $\mathcal{Z}_n$ should be asymptotic to $\pi^{-1}\kappa^2 n$. This is indeed true; see Section 2.7.

2.12 CURVES OF ZEROS

Beginning with this section, we investigate the behavior of the partition function $\mathcal{Z}_n(\beta)$ in infinitesimal windows located near the boundaries of the phases. Our results shed light on an interesting phenomenon, the *curves of zeros*, observed by Derrida [74] in the context of the REM.

BEAK SHAPED BOUNDARIES. We start by considering the boundary between two GE-phases of the form $G^{l-1}E^{d-l+1}$ and $G^{l}E^{d-l}$, where $1 \leq l \leq d$; see Figure 2.5. In phase $G^{l-1}E^{d-l+1}$, the fluctuations of $\mathcal{Z}_n(\beta)$ are given by an $(l-1)$-variate Poisson cascade zeta function, whereas in phase $G^{l}E^{d-l}$ the fluctuations are given by an l-variate zeta function; see Theorem 2.11.4. Near the boundary between these two phases, under an appropriate scaling, *both* functions become "visible" in the limit.

Theorem 2.12.1 (Fluctuations Near the Beak-shaped Boundaries Between GE-phases)**.** *Fix some* $1 \leq l \leq d$ *and some* $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ *such that* $\sigma_* > \frac{\sigma_l}{2}$, $\tau_* > 0$ *and* $\sigma_* + \tau_* = \sigma_l$*. Then, there exist a complex sequence* $d_{n,l} = O(\log n)$ *and a sequence of functions* $h_{n,l}(t)$ *(which are quadratic functions in* t*) such that weakly on* $\mathcal{H}(\mathbb{C})$ *it holds that*

$$\left\{ e^{-h_{n,l}(t)} \mathcal{Z}_n\left(\beta_* + \frac{d_{n,l} + t}{a_l(\beta_* - \sigma_l)n}\right) : t \in \mathbb{C}\right\} \xrightarrow[N\to\infty]{w} \left\{ e^t \zeta^{(l-1)} + \zeta^{(l)} : t \in \mathbb{C}\right\}. \tag{2.63}$$

Here, $(\zeta^{(l-1)}, \zeta^{(l)})$ is a random vector given by

$$(\zeta^{(l-1)}, \zeta^{(l)}) = \left(\zeta_P\left(T^{l-1}(\beta_*)\right), \zeta_P\left(T^{l}(\beta_*)\right)\right). \tag{2.64}$$

In (2.64), both $\zeta^{(l-1)}$ and $\zeta^{(l)}$ are based on the same Poisson cascade point process.

We will not provide exact expressions for $d_{n,l}$ and $h_{n,l}(t)$ (which are too complicated to be stated here). Theorem 2.12.1 allows us to clarify the local structure of the line of zeros near the beak shaped boundary between the phases $G^{l-1}E^{d-l+1}$ and $G^l E^{d-l}$.

Corollary 2.12.1 (Local distribution of zeros near the beak shaped boundaries between GE-phases)**.** *Under the same conditions as in Theorem 2.12.1, the following convergence of point processes holds weakly on $\mathcal{N}(\mathbb{C})$:*

$$\textbf{Zeros}\left\{\mathcal{Z}_n\left(\beta_* + \frac{d_{n,l}+t}{a_l(\beta_* - \sigma_l)n}\right) : t \in \mathbb{C}\right\} \underset{N\to\infty}{\overset{w}{\longrightarrow}} \sum_{k\in\mathbb{Z}} \delta\left(\log\left(-\frac{\zeta^{(l-1)}}{\zeta^{(l)}}\right) + 2\pi i k\right). \tag{2.65}$$

It follows that the zeros of $\mathcal{Z}_n(\beta)$ in a neighborhood of β_* look locally like *equally spaced* points on a line parallel to the boundary between $G^{l-1}E^{d-l+1}$ and $G^l E^{d-l}$; see Figure 2.6, right. The spacing between neighboring zeros is

$$\frac{\sqrt{2}\pi}{a_l \tau_*} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right), \quad n \to \infty. \tag{2.66}$$

This agrees with what one expects from the definition of the measure Ξ_l^{EG}; see Section 2.7. From the formula for $d_{n,l}$, it can be seen that the zeros are located *outside* the phase E_l, the distance to the boundary being of order const $\cdot \frac{\log n}{n}$. One may ask what happens if we drop the sequence $d_{n,l}$ and look at $\mathcal{Z}_n(\beta)$ in a window of size $\frac{1}{n}$ near some β_* located on the beak shaped boundary. The zeros are not visible in this regime and only the term $\zeta^{(l-1)}$ (the contribution of the inner phase $G^{l-1}E^{d-l+1}$) is present in the fluctuations.

CURVES OF ZEROS: ARC-SHAPED BOUNDARIES In the next theorem, we describe the local structure of the partition function $\mathcal{Z}_n(\beta)$ in an infinitesimal window around some $\beta_* = \sigma_* + i\tau_*$ located on the arc separating the phases $G^{d_1}F^{d_2}E^{d_3}$ and $G^{d_1}F^{d_2-1}E^{d_3+1}$, where $d_2 \geq 1$. We assume that

$$\frac{\sigma_{d_1}}{2} < \sigma_* < \frac{\sigma_{d_1+1}}{2}, \quad \tau_* > 0, \quad \sigma_*^2 + \tau_*^2 = \frac{\sigma_{d_1+d_2}^2}{2}. \tag{2.67}$$

Consider the following sequence of normalizing functions (which are linear in $t \in \mathbb{C}$):

$$\begin{aligned} f_n(\beta_*; t) := &\left(\beta_* + \frac{t}{n}\right) \sum_{j=1}^{d_1} \sqrt{n a_j} u_{n,j} + \\ &+ \sum_{j=d_1+1}^{d_1+d_2} \left(\frac{1}{2}\log N_{n,j} + a_j \sigma_*^2 n\right) + \sum_{j=d_1+d_2+1}^{d} \left(\log N_{n,j} + \frac{1}{2} a_j \beta_*^2 n\right). \end{aligned} \tag{2.68}$$

Theorem 2.12.2 (Fluctuations on the Arc-shaped Boundaries between Phases with F-levels)**.** *Fix some $d_1, d_2, d_3 \in \{0, \dots, d\}$ with $d_1 + d_2 + d_3 = d$ and $d_2 \geq 2$. Also, fix some $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ satisfying (2.67). Then, the following convergence of random analytic functions holds weakly on $\mathcal{H}(\mathbb{C})$:*

$$\left\{ \mathrm{e}^{-f_n(\beta_*;t)} \mathcal{Z}_n \left(\beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \underset{N\to\infty}{\overset{w}{\longrightarrow}} \left\{ \sqrt{W} \left(\mathrm{e}^{\lambda' t} N' + \mathrm{e}^{\lambda'' t} N'' \right) : t \in \mathbb{C} \right\}, \quad (2.69)$$

where

(1) *$W = \zeta_P(2T^{d_1}(\sigma_*))$, and ζ_P is the Poisson cascade zeta function with d_1 variables;*

(2) *$N', N'' \sim N_{\mathbb{C}}(0,1)$ are independent complex standard normal random variables;*

(3) *λ', λ'' are constants given in Remark 2.12.1 below;*

(4) *the random variable W and the random vector (N', N'') are independent.*

Remark 2.12.1. *Define the "partial variances" $A_{l,m} = a_l + \dots + a_m$ for $1 \leq l \leq m \leq d$ and let $A_{l,m} = 0$ if $l > m$. The constants λ' and λ'' are given by*

$$\begin{aligned} \lambda' &= 2\sigma_* A_{d_1+1,d_1+d_2} + \beta_* A_{d_1+d_2+1,d}, \\ \lambda'' &= 2\sigma_* A_{d_1+1,d_1+d_2-1} + \beta_* A_{d_1+d_2,d}. \end{aligned} \quad (2.70)$$

Note that $\lambda' - \lambda'' = \bar{\beta} a_{d_1+d_2}$.

Corollary 2.12.2 (Zeros on the Arc-shaped Boundaries between Phases with F-levels)**.** *Under the same conditions as in Theorem 2.12.2, we have the following weak convergence of point processes on $\mathcal{N}(\mathbb{C})$:*

$$\mathbf{Zeros} \left\{ \mathcal{Z}_n \left(\beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \underset{N\to\infty}{\overset{w}{\longrightarrow}} \sum_{k\in\mathbb{Z}} \delta \left(\frac{1}{\bar{\beta} a_{d_1+d_2}} \left(\log \left(-\frac{N''}{N'} \right) + 2\pi i k \right) \right). \quad (2.71)$$

In the case $d_2 = 1$, we have a slightly different result. Let $f_n(\beta_*; t)$ be given by the same formula as above; see (2.68).

Theorem 2.12.3 (Fluctuations on the Arc-shaped Boundaries of GE-phases)**.** *Fix some $d_1, d_2, d_3 \in \{0, \dots, d\}$ with $d_1 + d_2 + d_3 = d$ and $d_2 = 1$. Also, fix some $\beta_* = \sigma_* + i\tau_* \in \mathbb{C}$ satisfying (2.67). Then, the following convergence of random analytic functions holds weakly on $\mathcal{H}(\mathbb{C})$:*

$$\left\{ \mathrm{e}^{-f_n(\beta_*;t)} \mathcal{Z}_n \left(\beta_* + \frac{t}{n} \right) : t \in \mathbb{C} \right\} \underset{N\to\infty}{\overset{w}{\longrightarrow}} \left\{ \mathrm{e}^{\lambda' t} \sqrt{W} N + \mathrm{e}^{\lambda'' t} \zeta^{(d_1)} : t \in \mathbb{C} \right\}, \quad (2.72)$$

where

(1) *$W = \zeta_P(2T^{d_1}(\sigma_*))$ and $\zeta^{(d_1)} = \zeta_P(T^{d_1}(\beta_*))$, where in both cases the zeta function ζ_P is based on the same Poisson cascade point process;*

(2) *$N \sim N_{\mathbb{C}}(0,1)$ is a complex standard normal random variable;*

(3) *λ' and λ'' are constants given in Remark 2.12.1;*

(4) *the random vector* $(W, \zeta^{(d_1)})$ *and the random variable* N *are independent.*

Corollary 2.12.3 (Zeros on the Arc-shaped Boundaries of GE-phases)**.** *Under the same conditions as in Theorem 2.12.3, we have the following weak convergence of point processes on* $\mathcal{N}(\mathbb{C})$*:*

$$\mathbf{Zeros}\left\{\mathcal{Z}_n\left(\beta_* + \frac{t}{n}\right) : \beta \in \mathbb{C}\right\} \xrightarrow[N\to\infty]{w} \sum_{k\in\mathbb{Z}} \delta\left(\frac{1}{\bar{\beta}_* a_{d_1+1}}\left(\log\left(-\frac{\zeta^{(d_1)}}{\sqrt{WN}}\right) + 2\pi i k\right)\right). \tag{2.73}$$

In Corollaries 2.12.2 and 2.12.3, the zeros of $\mathcal{Z}_n(\beta)$ in a neighborhood of β_* look locally like equally spaced points, the spacing being

$$\frac{2\pi}{a_{d_1+d_2}|\beta_*|} \cdot \frac{1}{n} + o\left(\frac{1}{n}\right), \quad n \to \infty. \tag{2.74}$$

This agrees with what we expect from the definition of the measure $\Xi^{EF}_{d_1+d_2}$; see Section 2.7.

2.13 DISCUSSION, EXTENSIONS AND OPEN PROBLEMS

CONJECTURES ON FURTHER VARIATIONS OF THE MODEL. Similarly to the setup of Chapter 1, it is natural to consider a complex GREM with arbitrary correlations between the real and imaginary parts of the random exponents. That is, given correlation parameters $\rho_1, \dots, \rho_d \in [-1, 1]$, consider a Gaussian random field $\{Y_\varepsilon \colon \varepsilon \in \mathbb{S}_n\}$ having the same distribution as $\{X_\varepsilon \colon \varepsilon \in \mathbb{S}_n\}$, see (2.3), and satisfying

$$\mathrm{Cov}(X_\varepsilon, Y_\eta) = \sum_{k=1}^{l(\varepsilon,\eta)} \rho_k a_k, \qquad \varepsilon, \eta \in \mathbb{S}_n, \tag{2.75}$$

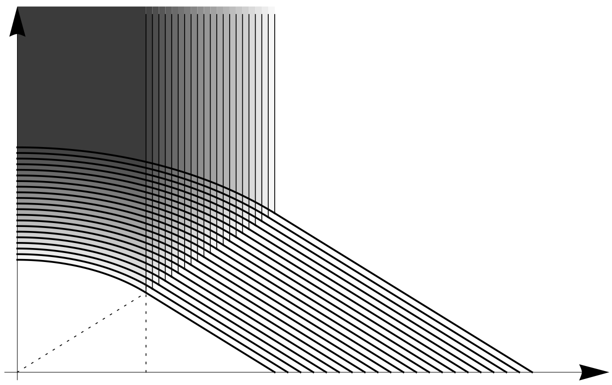
where $l(\varepsilon, \eta) = \min\{k \in \mathbb{N} \colon \varepsilon_k \neq \eta_k\} - 1$. Along the lines of the present chapter, one can study the partition function

$$\hat{\mathcal{Z}}_n(\beta) = \sum_{\varepsilon\in\mathbb{S}_n} \mathrm{e}^{\sqrt{n}(\sigma X_\varepsilon + i\tau Y_\varepsilon)}, \qquad \beta = (\sigma, \tau) \in \mathbb{R}^2. \tag{2.76}$$

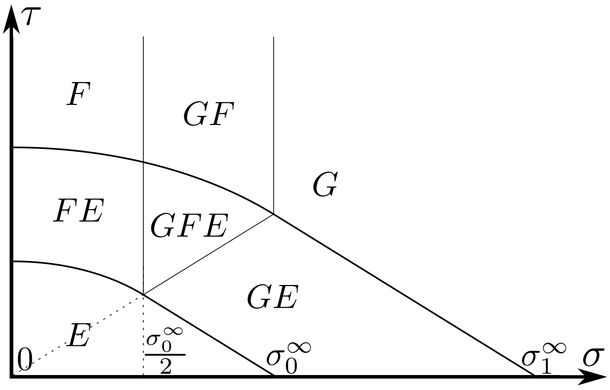
It seems that Theorems 2.4.1 and 2.7.1 need no changes even if we substitute partition function (2.9) with the one from (2.76). The more refined results on fluctuations such as Theorem 2.10.1, however, need appropriate modifications; see Chapter 1 for the case $d = 1$.

CONJECTURES ON MODELS WITH INFINITELY MANY HIERARCHIES. In this paragraph, we provide heuristics and state conjectures on the shapes of the phase diagrams for the models with *infinitely many hierarchies* ($d = \infty$).

In the GREM with d levels, there are d (real temperature) phase transitions at inverse temperatures $\beta = \sigma_1, \dots, \sigma_d$, whereas the spin glass models like the SK model are conjectured to exhibit a "continuum of freezing phase transitions" or the so-called *full replica symmetry breaking*, cf., [151, 164]. It has been suggested by Derrida and Gardner [77] that it is possible to consider the limit of the GREM as d, the number of levels, goes to ∞. Bovier and Kurkova [41] defined the limiting object for the GREM, the *Continuous Random Energy Model* (=CREM), and



Approximating CREM by GREM with many levels.



Conjectured phase diagram of the CREM w.r.t. complex temperatures.

Figure 2.8: (Conjectural) phase diagram of the CREM.

computed its free energy at real β. In this section, we will show heuristically how to pass to the infinite hierarchy limit of the GREM in the complex β case; see Figure 2.8.

Let $A\colon [0,1] \to \mathbb{R}$ be an increasing, concave function with $A(0) = 0$. Fix also some $\alpha > 1$. Consider a GREM with d levels whose parameters $(a_1, \ldots, a_d)$ and $(\alpha_1, \ldots, \alpha_d)$ are given by

$$a_1 + \ldots + a_k = A\left(\frac{k}{d}\right), \quad \log \alpha_k = \frac{1}{d} \log \alpha, \quad 1 \le k \le d. \tag{2.77}$$

The total number of energies in this GREM is $\alpha^{n+o(1)}$ and the variance of each energy is $A(1)n$.

Let us now pass to the large d limit. Let $t \in [0,1]$. Then, it follows from (2.77) that the large d limit of $da_{[td]}$ is $A'(t)$. Hence, the large d limit of the critical temperature $\sigma_{[td]}$ is

$$\sigma_t^\infty = \sqrt{\frac{2 \log \alpha}{A'(t)}}. \tag{2.78}$$

The large d limits of the domains $G_{[td]}$, $F_{[td]}$, $E_{[td]}$ are the domains

$$G_t^\infty := \{\beta \in \mathbb{C}\colon 2|\sigma| > \sigma_t^\infty, |\sigma| + |\tau| > \sigma_t^\infty\}, \tag{2.79}$$

$$F_t^\infty := \{\beta \in \mathbb{C}\colon 2|\sigma| < \sigma_t^\infty, 2(\sigma^2 + \tau^2) > (\sigma_t^\infty)^2\}, \tag{2.80}$$

$$E_t^\infty := \mathbb{C} \backslash \overline{G_t^\infty \cup F_t^\infty}. \tag{2.81}$$

Recall that the complex plane phases of a GREM with d levels were denoted by $G^{d_1} F^{d_2} E^{d_3}$, where the parameters $d_1, d_2, d_3 \in \mathbb{N}_0$ satisfy $d_1 + d_2 + d_3 = d$. Instead of d_1, d_2, d_3, in the continuum limit we have three parameters $\gamma_1, \gamma_2, \gamma_3 \in [0,1]$ which are the large d limits of $\frac{d_1}{d}, \frac{d_2}{d}, \frac{d_3}{d}$ and hence satisfy $\gamma_1 + \gamma_2 + \gamma_3 = 1$. To find the formula for $\gamma_1, \gamma_2, \gamma_3 \in [0,1]$ note that in the d-level GREM,

$$d_1 = \max\{k \ge 0\colon \beta \in G_k\}, \quad d_1 + d_2 + 1 = \max\{k \ge 0\colon \beta \in E_k\}. \tag{2.82}$$

Passing to the large d limit, we obtain

$$\gamma_1 = \sup\{t \in [0,1] : \beta \in G_t^\infty\}, \quad \gamma_1 + \gamma_2 = \sup\{t \in [0,1] : \beta \in E_t^\infty\}. \tag{2.83}$$

For the d-level GREM, Theorem 2.4.1 states that the log-partition function is $p(\beta) = p_G(\beta) + p_F(\beta) + p_E(\beta)$,where $p_G(\beta)$, $p_F(\beta)$, $p_E(\beta)$ are the contributions of glassy, fluctuation and expectation levels given by

$$p_G(\beta) := |\sigma| \sum_{k=1}^{d_1} \sqrt{2a_k \log \alpha_k}, \tag{2.84}$$

$$p_F(\beta) := \sum_{k=d_1+1}^{d_1+d_2} \left(\frac{1}{2} \log \alpha_k + a_k \sigma^2 \right), \tag{2.85}$$

$$p_E(\beta) := \sum_{k=d_1+d_2+1}^{d} \left(\log \alpha_k + \frac{1}{2} a_k (\sigma^2 - \tau^2) \right). \tag{2.86}$$

Replacing Riemann sums by Riemann integrals, we obtain that in the large d limit, the log-partition function of the CREM is

$$p^\infty(\beta) := p_G^\infty(\beta) + p_F^\infty(\beta) + p_E^\infty(\beta), \tag{2.87}$$

where

$$p_G^\infty(\beta) := |\sigma| \sqrt{2 \log \alpha} \int_0^{\gamma_1} \sqrt{A'(t)} \mathrm{d}t, \tag{2.88}$$

$$p_F^\infty(\beta) := \frac{\gamma_2}{2} \log \alpha + (A(\gamma_1 + \gamma_2) - A(\gamma_1))\sigma^2, \tag{2.89}$$

$$p_E^\infty(\beta) := \gamma_3 \log \alpha + \frac{1}{2}(\sigma^2 - \tau^2)(A(1) - A(\gamma_1 + \gamma_2)). \tag{2.90}$$

If β is real, then γ_1 is the solution of $\sigma_{\gamma_1}^\infty = \sigma$, $\gamma_2 = 0$, $\gamma_3 = 1 - \gamma_1$ and the log-partition function is given by

$$p^\infty(\beta) := |\sigma| \sqrt{2 \log \alpha} \int_0^{\gamma_1} \sqrt{A'(t)} \mathrm{d}t + (1 - \gamma_1) \log \alpha + \frac{\sigma^2}{2}(A(1) - A(\gamma_1)). \tag{2.91}$$

This formula is known, see [41, Theorem 3.3] (where the second term is missing) and [45, Theorem 4.2] (where all terms are present).

In the continuum limit of the GREM, we conjecture that there are seven phases

$$GFE, GF, FE, GE, G, F, E; \tag{2.92}$$

see Figure 2.8. In such a phase, for every letter which is not in the name of this phase, the corresponding γ_i must vanish. For example, the phase FE is characterized by the conditions $\gamma_1 = 0$, $\gamma_2 \neq 0$, $\gamma_3 \neq 0$.

It should be stressed that we have no rigorous proof that (2.87), (2.88), (2.89), (2.90) apply to the CREM as defined by Bovier & Kurkova [41]. In the real β case, Bovier and Kurkova [41] were able to sandwich a CREM between two close GREM's which allowed them to derive (2.91) rigorously using Gaussian comparison inequalities. This method does not seem to work in the complex β case because we cannot apply the comparison inequalities.

The branching random walk, branching Brownian motion and the Gaussian multiplicative chaos can be seen as the limiting cases of the CREM with $A(t) = t$. In this case, $\sigma_t^\infty \equiv 1$ which means that we have only the phases E, F, G as in the REM, see Derrida *et al.* [75], Lacoin *et al.* [133], Madaule *et al.* [144], and Madaule *et al.* [145] and Chapter 3.

3 COMPLEX BRANCHING BROWNIAN MOTION

What happens at the borderline of the REM universality class? Motivated by this question we consider *branching Brownian motion*, which is a representative of the class of the so-called log-correlated random fields. We show that this model lies exactly at the borderline of the REM universality class. We consider the partition function of the field of energies given by the "positions" of the particles of the *complex-valued* BBM. In such a *complex BBM energy model*, we allow for arbitrary correlations between the real and imaginary parts of the energies. We identify the fluctuations of the partition function. As a consequence, we get the full phase diagram of the log-partition function. It turns out that the phase diagram is the same as for the field of independent energies, i.e., Derrida's random energy model. Yet, the fluctuations are different from those of the REM in all phases. All results are shown *for any correlation* between the real and imaginary parts of the random energy.

This chapter is based on publications 5a., 6a.

In this chapter, we focus on the *complex-valued branching Brownian Motion energy model* and show that this model lies exactly at the borderline of the complex REM universality class. This means that the *phase diagram* of the model is the same as in the complex REM, Chapter 1. However, the fluctuations of the partition function of this model are already influenced by the strong correlations and differ from those of the REM in *all phases* of the model.

3.1 BRANCHING BROWNIAN MOTION.

BBM viewed as a random energy model plays a special rôle. It turns out that BBM has correlations which are exactly at the borderline between the regime of weak correlations (REM universality class[1]) and the one of strong correlations[2].

Before stating our results, let us briefly recall the construction of a BBM. Consider a canonical continuous branching process: a *continuous time Galton-Watson* (GW) process Athreya & Ney [13]. It starts with a single particle located at the origin at time zero. After an exponential time of parameter one, this particle splits into $k \in \mathbb{Z}_+$ particles according to some probability distribution $(p_k)_{k\geq 0}$ on $\mathbb{Z}_+$. Then, each of the new-born particles splits independently at

1 = the same phase diagram as for the field of independent random energies.
2 = different phase diagram comparing to the REM one, due to the strictly larger leading order of the minimal energy than the one for the independent field of random energies.

independent exponential (parameter 1) times again according to the same $(p_k)_{k\geq 0}$, and so on. We assume that $\sum_{k=1}^{\infty} p_k = 1$.[3] In addition, we assume that $\sum_{k=1}^{\infty} kp_k = 2$ (i.e., the expected number of children per particle equals two). Besides, we impose the finite second moment assumption:

$$K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty. \tag{3.1}$$

We assume that at time $t = 0$, the GW process starts with just one particle.

For given $t \geq 0$, we label the particles of the process as $i_1(t), \ldots, i_{n(t)}(t)$, where $n(t)$ is the total number of particles at time t. Note that under the above assumptions, we have $\mathbb{E}[n(t)] = e^t$. For $s \leq t$, we denote by $i_k(s,t)$ the unique ancestor of particle $i_k(t)$ at time s. In general, there will be several indices k, l such that $i_k(s,t) = i_l(s,t)$. For $s, r \leq t$, define the time of the most recent common ancestor of particles $i_k(r,t)$ and $i_l(s,t)$ as[4]

$$d(i_k(r,t), i_l(s,t)) := \sup\{u \leq s \wedge r \colon i_k(u,t) = i_l(u,t)\}. \tag{3.2}$$

For $t \geq 0$, the collection of all ancestors naturally induces the random tree

$$\mathbb{T}_t := \{i_k(s,t) \colon 0 \leq s \leq t, 1 \leq k \leq n(t)\} \tag{3.3}$$

called the *GW tree up to time t*. We denote by $\mathcal{F}^{\mathbb{T}_t}$ the σ-algebra generated by the GW process up to time t.

In addition to the genealogical structure, the particles get a *position* in $\mathbb{R}$. Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time when the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the $n(t)$ particles at time $t \geq 0$ by $x_1(t), \ldots, x_{n(t)}(t)$, see Figure 3.1.

We define BBM as a family of Gaussian processes,

$$x_t := \{x_1(s,t), \ldots, x_{n(t)}(s,t) \colon s \leq t\} \tag{3.4}$$

indexed by time horizon $t \geq 0$. Note that conditionally on the underlying GW tree these Gaussian processes have the following covariance

$$\mathbb{E}\left[x_k(s,t)x_l(r,t) \mid \mathcal{F}^{\mathbb{T}_t}\right] = d(i_k(s,t), i_l(r,t)), \quad s, r \in [0,t], \quad k, l \leq n(t). \tag{3.5}$$

In what follows, to lighten the notation, we will simply write $x_i(s) := x_i(s,t)$, $i \leq n(t)$, $s \leq t$ hoping that this will not cause confusion about the parameter $t \geq 0$.

3.2 A MODEL WITH COMPLEX-VALUED RANDOM ENERGIES

In this section, we introduce the *complex BBM random energy model*.

3 This implies that $p_0 = 0$, so none of the particles ever dies.
4 Note that $d(\cdot,\cdot)$ is not the distance to the most recent common ancestor of the particles but rather the *overlap* between the particle trajectories.

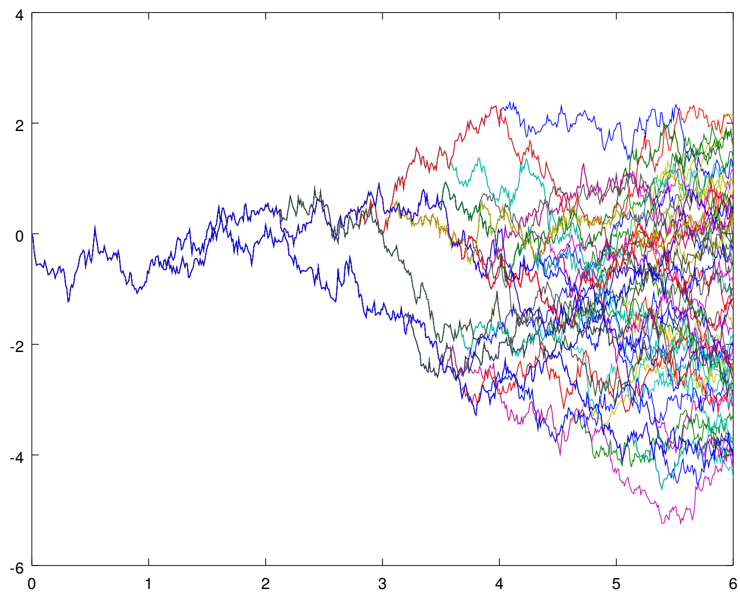


Figure 3.1: A simulated realization of a branching Brownian motion. Compare Figures 1.1 and 2.2

Let $\rho \in [-1,1]$. For any $t \in \mathbb{R}_+$, let $X(t) := (x_k(t))_{k \leq n(t)}$ and $Y(t) := (y_k(t))_{k \leq n(t)}$ be two BBMs with the same underlying GW tree such that, for $k \leq n(t)$,

$$\mathrm{Cov}(x_k(t), y_k(t)) = \rho t. \tag{3.6}$$

In what follows, to lighten the notation, we sometimes drop the dependence of quantities of interest on ρ. Note that

$$Y(t) \stackrel{\mathrm{D}}{=} \rho X(t) + \sqrt{1-\rho^2} Z(t), \tag{3.7}$$

where "$\stackrel{\mathrm{D}}{=}$" denotes equality in distribution and $Z(t) := (z_i(t))_{i \leq n(t)}$ is a branching Brownian motion independent from $X(t)$ and with the same underlying GW process. Representation (3.7) allows us to handle arbitrary correlations by decomposing the process Y into a part independent from X and a fully correlated one.

We define the *partition function* for the complex BBM energy model with correlation ρ at inverse temperature $\beta := \sigma + i\tau \in \mathbb{C}$ by

$$\mathcal{X}_{\beta,\rho}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t) + i\tau y_k(t)}. \tag{3.8}$$

NOTATION. By $\mathcal{L}[\cdot]$, $\mathcal{L}[\cdot \mid \cdot]$, and $\Longrightarrow$ or wlim, we denote the law, conditional law, and weak convergence respectively. By $\mathcal{N}(0, s^2)$, $s^2 > 0$, we denote the centred complex isotropic Gaussian distribution with density

$$\mathbb{C} \ni z \mapsto \frac{\mathrm{e}^{-|z/s|^2}}{\pi s^2} \in \mathbb{R}_+ \tag{3.9}$$

w.r.t. the Lebesgue measure on $\mathbb{C}$.

3.3 PHASE DIAGRAM

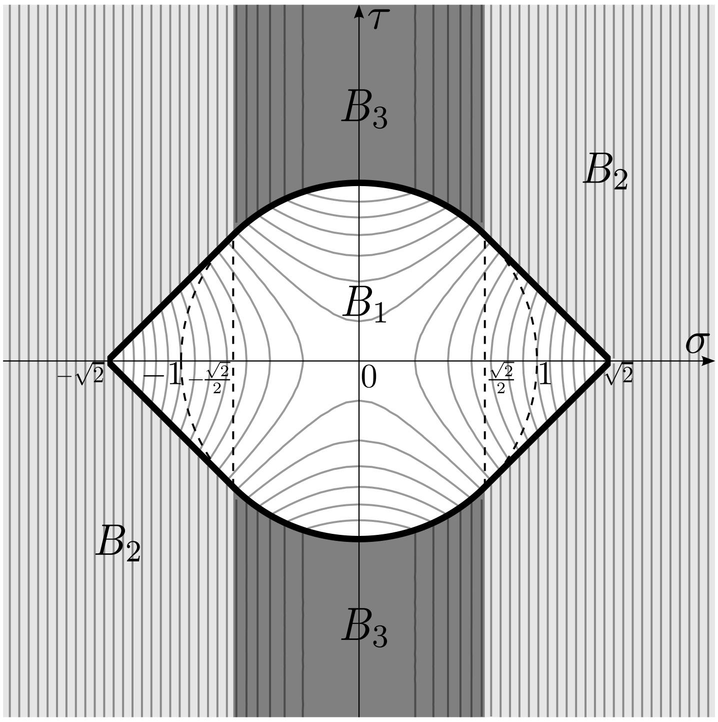


Figure 3.2: Phase diagram of the REM and the BBM energy model. The grey curves are the level lines of the limiting log-partition function, cf. (3.11).

Let us specify the three domains depicted on Figure 3.2 analytically:

$$B_1 := \mathbb{C} \setminus \overline{B_2 \cup B_3}, \quad B_2 := \{\sigma + i\tau \in \mathbb{C} : 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \\ B_3 := \{\sigma + i\tau \in \mathbb{C} : 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \tag{3.10}$$

Remark 3.3.1. *Some of our results will be stated under the binary branching assumption (i.e., $p_k = 0$ for all $k > 2$). Existence of all moments of the offspring distribution would also suffice for all our results and will not require essential changes in the proofs.*

Our first result states that the complex BBM energy model indeed has the phase diagram depicted on Figure 3.2.

Theorem 3.3.1 (Phase diagram). *For any $\rho \in [-1, 1]$, and any $\beta \in \mathbb{C}$, the complex BBM energy model with binary branching has the same log-partition function and the phase diagram (cf.,* Figure 3.2*) as the complex REM, i.e.,*

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \mathcal{X}_{\beta,\rho}(t) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B_1}, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B_2}, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B_3} \end{cases} \tag{3.11}$$

in probability.

See Section 2.6 for a proof.

Remark 3.3.2. 1. *For a deterministic regular weighted tree (a.k.a. directed polymer on a tree), under the assumption of no correlations between the real and imaginary parts of the complex random energies (i.e., case $\rho = 0$), formula (3.11) was obtained by Derrida et al. [75]. Our derivation of Theorem 3.3.1 is based on the detailed information on the fluctuations of the partition function (3.8) which we provide in Section 3.5. The proof in [75] is more direct and does not reach the (CLT) precision which is provided in the following section. Moreover, the arguments in [75] seem to crucially rely on the assumption $\rho = 0$.*

2. *It is natural to expect that the convergence in (3.11) also holds in L^1, see Theorem 1.4.1 for a related result for the REM.*

3.4 GLASSY PHASE B_2

In this section, we focus on the *glassy phase* B_2. In this phase, the main contribution to the partition function comes from the particles with the largest real parts of the energy. Let us recall some of known facts about the *extremal particles* of the branching Brownian motin.

Bramson [46] and Bramson [47] showed that

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t \tag{3.12}$$

is the order of the maximal position among all BBM particles alive at large time t, i.e.,

$$\lim_{t\uparrow\infty} \mathbb{P}\left\{ \max_{k\le n(t)} x_k(t) - m(t) \le y \right\} = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \quad y \in \mathbb{R}, \tag{3.13}$$

where $C > 0$ is a constant and Z is the a.s. limit of the so-called *derivative martingale*:

$$Z := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t))\mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}, \quad \text{a.s.} \tag{3.14}$$

In Aïdékon *et al.* [1] and Arguin *et al.* [10], as $t \uparrow \infty$, the non-trivial limiting point process of the (shifted by $m(t)$) particles of BBM was identified. Specifically, it was shown that the point process,

$$\mathcal{E}_t := \sum_{k=1}^{n(t)} \delta_{x_k(t) - m(t)}, \quad t \in \mathbb{R}_+ \tag{3.15}$$

converges in law as $t \uparrow \infty$ to the point process (see Figure 3.4)

$$\mathcal{E} := \sum_{k,l} \delta_{\eta_k + \Delta_l^{(k)}}, \tag{3.16}$$

where:

(a) $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of a Cox-Poisson point process with *random intensity measure* $CZ\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$, where C and Z are the same as in (3.13). See Figure 3.3.

Figure 3.3: Poisson point process

(b) $\{\Delta_l^{(k)}\}_{l\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, $k \in \mathbb{N}$ called *clusters* which are independent copies of the limiting point process

$$\Delta := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\hat{x}_k(t) - \max_{l\le n(t)} \hat{x}_l(t)} \tag{3.17}$$

with $\hat{x}(t)$ being BBM $x(t)$ conditioned on $\max_{k\le n(t)} x_k(t) \ge \sqrt{2}t$.

Poisson point process + Cluster Δ_1

Figure 3.4: Poisson point process + all clusters

We start with the convergence of the partition function in the case of the real BBM energy model at complex temperatures. We say that a complex-valued r.v. Y is *isotropic α-stable* if there exists $c \in \mathbb{R}_+$ and $\alpha \in (0,2]$ such that

$$\mathbb{E}[e^{i\mathrm{Re}(\bar{z}Y)}] = e^{-c|z|^\alpha}, \quad \text{for all } z \in \mathbb{C}. \tag{3.18}$$

Recall the notation from (3.16).

Theorem 3.4.1 (Partition function fluctuations for $|\rho| = 1$)**.** *For $\beta = \sigma + i\tau \in B_2$, the rescaled partition function $\mathcal{X}_{\beta,1}(t) := e^{-\beta m(t)}\widetilde{\mathcal{X}}_{\beta,1}(t)$ converges in law to the r.v.*

$$\mathcal{X}_{\beta,1} := \sum_{k,l\ge1} e^{\beta\left(\eta_k+\Delta_l^{(k)}\right)}, \quad \textit{as } t\uparrow\infty. \tag{3.19}$$

For $|\rho| \in (0,1)$, we get the following result.

Theorem 3.4.2 (Partition function fluctuations for $|\rho| \in (0,1)$)**.** *For $\beta = \sigma + i\tau \in B_2$ and $|\rho| \in (0,1)$, the rescaled partition function $\mathcal{X}_{\beta,\rho}(t) := e^{-\sigma m(t)}\widetilde{\mathcal{X}}_{\beta,\rho}(t)$ converges in law to the r.v. $\mathcal{X}_{\beta,\rho}$, as $t\uparrow\infty$. Conditionally on Z, $\mathcal{X}_{\beta,\rho}$ is a complex isotropic $\sqrt{2}/\sigma$-stable r.v.*

Remark 3.4.1. *For $\rho = 0$, Theorem 3.4.2 was proven in Madaule* et al. *[144]. Our proof uses a representation of correlated real and imaginary parts in terms of independent BBM's. As in Madaule* et al. *[144], we control second moments. However, the way we do this is different and simpler then the method used in that paper, which relies on decomposing the paths of the BBM particles according to the time and location of the minimal position along the given path. Our approach uses instead the upper envelope for ancenstral paths that was obtained in Arguin* et al. *[12].*

3.5 A CLASS OF MARTINGALES

In the centre of our analysis are the following martingales

$$\mathcal{M}_{\sigma,\tau}(t) := \mathrm{e}^{-t\left(1+2i\rho\sigma\tau+\frac{\sigma^2-\tau^2}{2}\right)}\mathcal{X}_{\beta,\rho}(t) = \sum_{k=1}^{n(t)} \mathrm{e}^{-t\left(1+2i\rho\sigma\tau+\frac{\sigma^2-\tau^2}{2}\right)}\mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)}. \tag{3.20}$$

We denote by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ the natural filtration associated to $(\mathcal{M}_{\sigma,\tau}(t))_{t\in\mathbb{R}_+}$.

Note that, for $\beta = \sigma \in [0, 1/\sqrt{2})$, $\mathcal{M}_{\sigma,0}(t)$ coincides with the McKean martingale introduced in Bovier & Hartung [38], where it was proven that these martingales converge almost surely and in L^1 to a non-degenerate limit.

The next theorem states that for $\beta \in B_1$ the martingales $\mathcal{M}_{\sigma,\tau}(t)$ are in L^p for some $p > 1$.

Theorem 3.5.1 (L^p martingale convergence in B_1)**.** *For $\beta = \sigma + i\tau$ with $\beta \in B_1, |\beta| \geq 1$, and any $\rho \in [-1,1]$, $\mathcal{M}_{\sigma,\tau}(t)$ is a martingale with expectation 1 and it is in L^p for $p \leq \frac{\sqrt{2}}{\sigma}$. Hence, the limit*

$$\lim_{t\uparrow\infty} \mathcal{M}_{\sigma,\tau}(t) =: \mathcal{M}_{\sigma,\tau} \tag{3.21}$$

exists a.s., in L^1, and is non-degenerate.

Proposition 3.5.1. *For $\beta \in \mathbb{C}$ with $|\beta| < 1$, $\mathcal{M}_{\sigma,\tau}(t)$ is an L^2-bounded martingale with expectation one. In particular, $\mathcal{M}_{\sigma,\tau}(t)$ converges to a non-degenerate limit $M\mathcal{M}_{\sigma,\tau}$ a.s. and in L^2 as t tends to infinity.*

On the boundary $B_{1,2}$ between phases B_1 and B_2, i.e., on the set

$$B_{1,2} := \overline{B_1} \cap \overline{B_2} = \{\sigma + i\tau \in \mathbb{C} \colon |\sigma| > 1/\sqrt{2}, |\sigma| + |\tau| = \sqrt{2}\} \tag{3.22}$$

a similar result still holds.

Theorem 3.5.2 (L^p martingale convergence on $B_{1,2}$)**.** *For $\beta \in B_{1,2}$ and any $\rho \in [-1,1]$, we have that $\mathcal{M}_\beta(t)$ is a L^p-bounded martingale, for any $p < \sqrt{2}/\sigma$ with expectation 1. Hence, the limit*

$$\lim_{t\uparrow\infty} \mathcal{M}_{\sigma,\tau}(t) =: \mathcal{M}_{\sigma,\tau} \tag{3.23}$$

exists a.s. in L^1, and is non-degenerate.

Remark 3.5.1. *Similar result for $\rho = 0$ has been obtained for the complex Gaussian multiplicative chaos in [133, Theorem 3.11].*

Remark 3.5.2 (Smoothing transforms)**.** *Note that the martingales $\mathcal{M}_{\sigma,\tau}(t)$ satisfy a recursive equation of the form*

$$\mathcal{L}\left[\mathcal{M}_{\sigma,\tau}(t+r)\right] = \mathcal{L}\Big[\sum_{k=1}^{n(r)} a_k(r)\mathcal{M}_{\sigma,\tau}^{(k)}(t)\Big], \tag{3.24}$$

where $\mathcal{M}_{\sigma,\tau}^{(k)}(t-r)$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}(t)$ and $a_k(r) \in \mathbb{C}$ are some complex weights independent from $\mathcal{M}_{\sigma,\tau}^{(k)}(t-r), k \in \mathbb{N}$. If a limit $\mathcal{M}_{\sigma,\tau}$ as $t \uparrow \infty$ of $\mathcal{M}_{\sigma,\tau}(t+r)$ exists, then it would have to satisfy the equation

$$\mathcal{L}\left[\mathcal{M}_{\sigma,\tau}\right] = \mathcal{L}\Big[\sum_{k=1}^{n(r)} a_k(r)\mathcal{M}_{\sigma,\tau}^{(k)}\Big], \tag{3.25}$$

where $\mathcal{M}_{\sigma,\tau}^{(k)}$ are i.i.d. copies of $\mathcal{M}_{\sigma,\tau}$. This type of equation is called complex smoothing transform. We refer to Meiners and Mentemeier [148] and Kolesko and Meiners [129] for more details.

3.6 CONDITIONAL CENTRAL LIMIT THEOREMS

The following three results cover the whole strip $|\sigma| < 1/\sqrt{2}$ and basically are "central limit theorems" (CLTs) with random variance.

Theorem 3.6.1 (CLT with random variance for $|\sigma| < 1/\sqrt{2}$, $\beta \in B_1$)**.** *Let $\beta = \sigma + i\tau$ with $|\sigma| < 1/\sqrt{2}$ and $\rho \in [-1,1]$. For $\beta \in B_1$,*

$$\operatorname{wlim}_{r\uparrow\infty} \operatorname{wlim}_{t\uparrow\infty} \mathcal{L}\left[\frac{\mathcal{M}_{\sigma,\tau}(t+r) - \mathcal{M}_{\sigma,\tau}(r)}{\mathrm{e}^{r(1-\sigma^2-\tau^2)}} \,\middle|\, \mathcal{F}_r\right] = \mathcal{N}\left(0, C_1 \mathcal{M}_{2\sigma,0}\right), \tag{3.26}$$

where $C_1 > 0$ is some constant.

Remark 3.6.1.
1. *The scaling on the l.h.s. of (3.26) does not depend on ρ.*
2. *The appearance of the random variance in Theorem 3.6.1 (and in the subsequent ones) is in sharp contrast with the REM (Chapter 1) and generalized REM (Chapter 2), where CLTs with deterministic variance hold for β in the strip $|\sigma| < 1/\sqrt{2}$.*
3. *For $\beta \in \mathbb{R}$, a result resembling Theorem 3.6.1 was obtained by Iksanov and Kabluchko in [116].*
4. *For a logarithmically correlated field of complex-valued random energies on a Euclidean space without correlations between the real and imaginary parts of the energy (i.e., case $\rho = 0$), a similar result was shown by Lacoin et al. [133, Theorem 3.1].*

Theorem 3.6.2 (CLT with random variance in B_3)**.** *For $\beta \in B_3$, $\rho \in [-1,1]$ and binary branching,*

$$\mathcal{L}\left[\frac{\mathcal{X}_{\beta,\rho}(t)}{\mathrm{e}^{t(1/2+\sigma^2)}} \,\middle|\, \mathcal{M}_{2\sigma,0}\right] \underset{t\uparrow\infty}{\Longrightarrow} \mathcal{N}\left(0, C_2 \mathcal{M}_{2\sigma,0}\right), \tag{3.27}$$

where $C_2 > 0$ is some constant.

Remark 3.6.2. *In case $\rho = 0$, a similar result has been obtained by Lacoin et al. [133, Theorem 4.2].*

A similar result also holds on the boundary between phases B_1 and B_3, i.e., on the set

$$B_{1,3} := \overline{B_1} \cap \overline{B_3} = \{\sigma + i\tau \in \mathbb{C} \colon \sigma^2 + \tau^2 = 1, |\sigma| < 1/\sqrt{2}\}. \tag{3.28}$$

Theorem 3.6.3 (CLT with random variance on $B_{1,3}$)**.** *For $\beta \in B_{1,3}$, $\rho \in [-1,1]$, and binary branching,*

$$\mathcal{L}\left[\frac{\mathcal{X}_{\beta,\rho}(t)}{\sqrt{t}\mathrm{e}^{t(1/2+\sigma^2)}} \,\middle|\, \mathcal{M}_{2\sigma,0}\right] \underset{t\uparrow\infty}{\Longrightarrow} \mathcal{N}\left(0, C_3 \mathcal{M}_{2\sigma,0}\right), \tag{3.29}$$

where $C_3 > 0$ is some constant.

Remark 3.6.3. *For $\rho = 0$, a similar result for Gaussian multiplicative chaos was obtained by Lacoin et al. [133, Theorem 4.2].*

Recall that the behaviour of the partition function at $\beta = \sqrt{2}$ is determined by the martingale $\mathcal{M}_{1,0}(t)$, which is related to another martingale – the so-called *derivative martingale* $\mathcal{Z}(t)$:

$$\mathcal{Z}(t) := \sum_{i=1}^{n(t)} (\sqrt{2}t - x_k(t)) \mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}. \tag{3.30}$$

Lalley and Sellke proved in [134] that $\mathcal{Z}(t)$ converges a.s. as $t \to \infty$ to a non-trivial limit $\mathcal{Z}$ which is a positive and a.s. finite random variable.

At the boundary,

$$B_{2,3} := \overline{B_2} \cap \overline{B_3} = \left\{ \sigma + i\tau \in \mathbb{C} \colon |\sigma| = 1/\sqrt{2}, |\tau| \geq 1/\sqrt{2} \right\}, \tag{3.31}$$

including the *triple point*

$$\beta_{1,2,3} := \overline{B_1} \cap \overline{B_2} \cap \overline{B_3} = (1+i)/\sqrt{2}, \tag{3.32}$$

after appropriate rescaling, we have the following CLT with random variance.

Theorem 3.6.4 (CLT with random variance for $|\sigma| = 1/\sqrt{2}$)**.** *Let $\beta = \sigma + i\tau$ with $|\sigma| = 1/\sqrt{2}$ and $\rho \in [-1, 1]$ and assume binary branching. Then:*

(i) For $\tau > 1/\sqrt{2}$,

$$\operatorname{wlim}_{r\uparrow\infty} \operatorname{wlim}_{t\uparrow\infty} \mathcal{L}\left[r^{1/4} \cdot \frac{\mathcal{X}_{\beta,\rho}(t+r)}{\mathrm{e}^{(t+r)(1/2+\sigma^2)}} \;\middle|\; \mathcal{F}_r \right] = \mathcal{N}\left(0, C_2 \sqrt{\frac{2}{\pi}} \mathcal{Z}\right). \tag{3.33}$$

(ii) For $\tau = 1/\sqrt{2}$,

$$\operatorname{wlim}_{r\uparrow\infty} \operatorname{wlim}_{t\uparrow\infty} \mathcal{L}\left[\frac{r^{1/4}}{\sqrt{t}} \cdot \frac{\mathcal{X}_{\beta,\rho}(t+r)}{\mathrm{e}^{(t+r)(1/2+\sigma^2)}} \;\middle|\; \mathcal{F}_r \right] = \mathcal{N}\left(0, C_3 \sqrt{\frac{2}{\pi}} \mathcal{Z}\right). \tag{3.34}$$

Remark 3.6.4. *For $\rho = 0$, a similar result for Gaussian multiplicative chaos was obtained by Lacoin et al. [133, Theorem 4.3].*

3.7 SUMMARY AND OUTLOOK

For the complex BBM energy model, we now know:

- *fluctuations* of the partition function;
- distribution of complex *zeros* of the partition function;
- the limiting *log-partition function*;
- the *phase diagram*.

Here are some open problems related to this chapter:

- Study the log-partition function of the *randomized ζ-function* at complex temperatures, see, e.g., Arguin *et al.* [9] for the model.
- Study complex plane phase diagrams of models with *microscopic interactions* such as the one from Chapter 4.

4 HIGH-DIMENSIONAL RUGGED LANDSCAPES SEEN THROUGH SPIN GLASSES

Finding the (space-height) distribution of the (local) extrema of high-dimensional strongly correlated random fields is a notorious hard problem with many applications. Following Fyodorov & Sommers [93], we focus on the *Gaussian fields with isotropic increments* and take the viewpoint of statistical physics. By exploiting various probabilistic symmetries, we rigorously derive the Fyodorov-Sommers formula for the log-partition function in the high-dimensional limit. The formula suggests a rich picture for the distribution of the local extrema akin to the celebrated spherical Sherrington-Kirkpatrick model with mixed p-spin interactions.

This chapter is based on publication 2a.

4.1 INTRODUCTION

In this chapter, we focus on the energy based model where the energy function is given by an arbitrary high-dimensional Gaussian field with isotropic increments.

Consider the *Gaussian random field* with *isotropic increments* $X = X_N = \{X_N(u) : u \in \mathbb{R}^N\}$, $N \in \mathbb{N}$. The adjective "isotropic" means here that the law of the increments of the field X is invariant under *rigid motions* (= translations and rotations) in $\mathbb{R}^N$. We are interested in the case $N \gg 1$ and in the case of *strongly correlated fields with high-dimensional correlation structure*. Therefore, we assume that the field X_N satisfies

$$\mathbb{E}\left[(X_N(u) - X_N(v))^2\right] = D\left(\frac{1}{N}\|u-v\|_2^2\right) =: D_N(\|u-v\|_2^2), \quad u,v \in \mathbb{R}^N, \tag{4.1}$$

where $\|\cdot\|_2$ denotes the *Euclidean norm* on $\mathbb{R}^N$ and the *correlator* $D\colon \mathbb{R}_+ \to \mathbb{R}_+$ is any admissible function. Complete characterization of all correlators D that are admissible in (4.1), for all N, is known, see Theorem 4.2.1. Note that the law of the field X_N is determined by (4.1) only up to an additive shift by a Gaussian random variable. In what follows, without loss of generality, we assume that $X_N(0) = 0$.

The model was heuristically analyzed in detail using the non-rigorous replica method by Fyodorov & Le Doussal [91] and Fyodorov & Sommers [93] in the case of the rotationally invariant configuration space:

$$S_N := \{u \in \mathbb{R}^N : \|u\|_2 \le L\sqrt{N}\}, \quad N \in \mathbb{N}. \tag{4.2}$$

Related model was previously suggested and analyzed in the physics literature by Mézard & Parisi [150].

In this chapter, we consider product spaces, which are not rotationally invariant:

$$S_N := S^N, \quad S \subset \mathbb{R}. \tag{4.3}$$

Let $\mu \in \mathcal{M}_{\text{finite}}(S)$ be such that the origin is contained in the interior of the convex hull of the support of μ. Define $\mu_N := \mu^{\otimes N} \in \mathcal{M}_{\text{finite}}(S_N)$. A canonical example is the *discrete hypercube* $S_N := \{-1;1\}^N$ equipped with the uniform a priori measure, i.e., $\mu(\{u\}) := 2^{-N}$, for all $u \in S_N$.

MAIN OBJECTS. We are interested in the asymptotic behavior of the *extremes* of the random field X_N on the sequence of the *particle state spaces* $S_N \subset \mathbb{R}^N$ as $N \uparrow +\infty$. The state spaces are assumed to be equipped with a sequence of *a priori reference measures* $\{\mu_N \in \mathcal{M}_{\text{finite}}(S_N) \mid N \in \mathbb{N}\}$. We now define the main quantities of interest in this work. Consider the *partition function*

$$Z_N(\beta) := \int_{S_N} \mu_N(\mathrm{d}u) \exp\left(\beta\sqrt{N} X_N(u)\right), \quad \beta \in \mathbb{R}. \tag{4.4}$$

We view (4.4) as an exponential functional of the field X_N, which is parametrized by the *inverse temperature* β. Heuristically, for large β (i.e., $\beta \uparrow +\infty$), the maxima of the field X_N give substantial contribution to the integral (4.4). The N-scalings in (4.4), (4.1) and the "size" of S_N are tailored for studying the large-N limit of the random *log-partition function*:

$$p_N(\beta) := \frac{1}{N} \log Z_N(\beta), \quad \beta \in \mathbb{R}. \tag{4.5}$$

For comparison with the theoretical physics literature, let us note that there one conventionally substitutes $\beta \mapsto -\beta$ in (4.4) (this has no effect on the distribution of Z_N due to the symmetry of the centered Gaussian distribution of the field X_N), and considers instead of (4.5) the *free energy*

$$f_N(\beta) := -\frac{1}{\beta} p_N(\beta), \quad \beta \in \mathbb{R}_+. \tag{4.6}$$

4.2 LONG AND SHORT-RANGE CORRELATIONS

Before stating our results let us recall the full classification of admissible in (4.1) correlators.

We recall some facts about high-dimensional Gaussian processes with isotropic increments. The following result can be extracted from the work [189] of A.M. Yaglom (see also [190]).

Theorem 4.2.1. *If X is a Gaussian random field with isotropic increments that satisfies (4.1), then one of the following two cases holds:*

1. ***[Isotropic field]** There exists the* correlation function $B\colon \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$\mathbb{E}\left[X_N(u)X_N(v)\right] = B\left(\frac{1}{N}\|u-v\|_2^2\right), \quad u, v \in \Sigma_N, \tag{4.7}$$

where the function B has the representation

$$B(r) = c_0 + \int_0^{+\infty} \exp\left(-t^2 r\right) \nu(\mathrm{d}t), \tag{4.8}$$

where $c_0 \in \mathbb{R}_+$ is a constant and $\nu \in \mathcal{M}_{\text{finite}}(\mathbb{R}_+)$ is a non-negative finite measure. In this case, the function D in (4.1) is expressed in terms of the correlation function B as

$$D(r) = 2(B(0) - B(r)). \tag{4.9}$$

2. ***[Non-isotropic field with isotropic increments]*** *The function D in (4.1) has the following representation*

$$D(r) = \int_0^{+\infty} \left[1 - \exp\left(-t^2 r\right)\right] \nu(\mathrm{d}t) + A \cdot r, \quad r \in \mathbb{R}_+, \tag{4.10}$$

where $A \in \mathbb{R}_+$ is a constant and $\nu \in \mathcal{M}((0;+\infty))$ is a σ-finite measure with

$$\int_0^{+\infty} \frac{t^2 \nu(\mathrm{d}t)}{t^2+1} < \infty. \tag{4.11}$$

Remark 4.2.1. *In Theorem 4.2.1, assuming $c_0 = 0$, case 1 is sometimes referred to as the* short-range *one which reflects the decay of correlations: $B(r) \downarrow +0$, as $r \uparrow +\infty$. This fact follows from the representation (4.8). Correspondingly, case 2 is called the* **long-range** *one, since here, assuming $X(0) = 0$, the correlation structure is*

$$\mathbb{E}\left[X_N(u) X_N(v)\right] = \frac{1}{2}\left(D_N(\|u\|_2^2) + D_N(\|v\|_2^2) - D_N(\|u-v\|_2^2)\right), \quad u, v \in \mathbb{R}^N. \tag{4.12}$$

Equation (4.12) in combination with the representation (4.10) implies that the correlations of the field X_N do not decay, as $\|u-v\| \to +\infty$.

Remark 4.2.2. *Theorem 4.2.1 implies that the function D appearing in (4.1) is necessarily concave, infinitely differentiable, and non-decreasing on $(0;+\infty)$.*

4.3 RESULTS

To formulate our results on the limiting log-partition function, we need the following definitions.

PARISI-TYPE FUNCTIONAL. Given $r \in \mathbb{R}_+$, consider the space of the *functional order parameters*

$$\mathcal{X}(r) := \{x\colon [0;r] \to [0;1] \mid x \text{ is non-decreasing càdlàg}, x(0) = 0, x(r) = 1\}, \tag{4.13}$$

It is convenient to work with the space of the *discrete order parameters*

$$\mathcal{X}'_n(r) := \{x \in \mathcal{X}(r) \mid x \text{ is piece-wise constant with at most } n \text{ jumps}\}. \tag{4.14}$$

Let us denote the *effective size* of the particle state space by

$$d := \sup_N \left(\frac{1}{N} \sup_{u \in S_N} \|u\|_2^2 \right). \tag{4.15}$$

For what follows, it is enough to assume that $r \in [0; d]$ in (4.13). Note that, in case (4.3), $d = \sup_{u \in S} u^2$.

Now, let us define the non-linear functional that appears in the variational formula of our main result. We do it in three steps:

1. Given large enough $M \in \mathbb{R}_+$, define the regularized derivative $D'^{,M} : \mathbb{R}_+ \to \mathbb{R}$ of the correlator D as

$$D'^{,M}(r) := \begin{cases} D'(r), & r \in [1/M; +\infty), \\ M, & r \in [0; 1/M). \end{cases} \tag{4.16}$$

Given $r, M \in \mathbb{R}_+$, define the function $\theta_r^{(M)} \colon [-r; r] \to \mathbb{R}$ as

$$\theta_r^{(M)}(q) := q D'^{,M}(2(r-q)) + \frac{1}{2} D(2(r-q)), \quad q \in [-r; r]. \tag{4.17}$$

2. Given $r \in \mathbb{R}_+$, $x \in \mathcal{X}(r)$ and the (sufficiently regular) *boundary condition* $h \colon \mathbb{R} \to \mathbb{R}$, consider the semi-linear parabolic *Parisi's terminal value problem*:

$$\begin{cases} \partial_q f(y,q) + \frac{1}{2} D'^{,M}(2(r-q)) \left(\partial^2_{qq} f(y,q) + x(q) \left(\partial_y f(y,q) \right)^2 \right) = 0, & (y,q) \in \mathbb{R} \times (0,r), \\ f(y,1) = h(y), & y \in \mathbb{R}. \end{cases} \tag{4.18}$$

Let $f_{r,x,h}^{(M)} \colon [0;1] \times \mathbb{R}_+ \to \mathbb{R}$ be the unique solution of (4.18). Solubility of the Parisi terminal value problem (4.18), its relation to the Hamilton-Jacobi-Bellman equations and stochastic control problems is discussed in a more general multidimensional context in [44, Section 6].

3. Given the family of the (sufficiently regular for (4.18) to be solvable) *boundary conditions*

$$g := \{ g_\lambda \colon \mathbb{R} \to \mathbb{R} \mid \lambda \in \mathbb{R} \}, \tag{4.19}$$

and given $r \in [0; d]$, define the *local Parisi functional* $\mathcal{P}(\beta, r, g) \colon \mathcal{X}(r) \to \mathbb{R}$ as

$$\mathcal{P}(\beta, r, g)[x] := \lim_{M \uparrow +\infty} \left(\inf_{\lambda \in \mathbb{R}} \left[f_{r,x,g_\lambda}^{(M)}(0,0) - \lambda r \right] - \frac{\beta^2}{2} \int_0^1 x(q) \mathrm{d}\theta_r^{(M)}(q) \right), \quad x \in \mathcal{X}(r). \tag{4.20}$$

In (4.20), the integral with respect to $\theta_r^{(M)}$ is understood in the Lebesgue-Stieltjes sense.

MAIN RESULTS. Let us start by recording the basic convergence result for the log-partition function.

Theorem 4.3.1 (Existence of the limiting free energy)**.** *For any $\beta > 0$, the large N-limit of the log-partition function exists and is a.s. deterministic:*

$$p_N(\beta) \underset{N\uparrow+\infty}{\longrightarrow} p(\beta), \quad \text{almost surely and in } L^1. \tag{4.21}$$

In addition, for any $N \in \mathbb{N}$, the following concentration of measure inequality holds

$$\mathbb{P}\left\{|p_N(\beta) - \mathbb{E}\left[p_N(\beta)\right]| > t\right\} \leq 2\exp\left(-\frac{Nt^2}{4D(d)}\right), \quad t \in \mathbb{R}_+. \tag{4.22}$$

The main result of this work is the following variational representation for the limiting log-partition function in terms of the Parisi functional (4.20).

Theorem 4.3.2 (Free energy variational representation, comparison with cascades)**.** *Assume* (4.3)*. Let the family of boundary conditions* (4.19) *be defined as*

$$g_\lambda(y) := \log \int_S \mu(\mathrm{d}u)\exp\left(\beta u y + \lambda u^2\right), \quad y \in \mathbb{R}. \tag{4.23}$$

Then, for all $\beta \in \mathbb{R}$,

$$p(\beta) := \sup_{r\in[0;d]} \inf_{x\in\mathcal{X}(r)} \left(\mathcal{P}(\beta, r, g)[x]\right), \quad \text{almost surely and in } L^1, \tag{4.24}$$

Remark 4.3.1. *In the case* (4.10)*, the field* (4.26) *has a feature, which is not within the assumptions typically found in the literature Guerra* [104]*, Guerra & Toninelli* [105]*, Panchenko* [165]*, Talagrand* [184]*, and Talagrand* [186]*: the correlator D is not of class C^1, namely, D can have a singular derivative at* 0*. To deal with the singularity, we need a regularization procedure, cf.* (4.16) *and* (4.20)*.*

HEURISTICS. It is natural to ask the following questions: Why is Parisi's theory of hierarchical replica symmetry breaking Mézard *et al.* [151] (which is usually behind the functionals of the type (4.20)) applicable to Gaussian fields with isotropic increments satisfying (4.1)? Where are the "interacting spins" in the present context?

A hint is given by the following observation. Define

$$\langle u, v\rangle_N := \frac{1}{N}\sum_{i=1}^{N} u_i v_i, \quad u, v \in \mathbb{R}^N. \tag{4.25}$$

Let us fix $r \in [0; d]$. By (4.12), the restriction of the field X_N with isotropic increments to a sphere with radius r centered at the origin, leads to the *mixed p-spin spherical SK model* (cf. [184]) with the following covariance structure

$$\mathbb{E}\left[X_N(u)X_N(v)\right] = D(r) - \frac{1}{2}D(2(r - \langle u, v\rangle_N)) =: G_r(\langle u, v\rangle_N), \quad \|u\|_2^2 = \|v\|_2^2 = rN, \tag{4.26}$$

where $G_r : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$G_r(q) := D(r) - \frac{1}{2}D(2(r - q)), \quad q \in \mathbb{R}_+. \tag{4.27}$$

Thus, (4.26) implies that, given r, each field of the type (4.1) induces a mixed p-spin spherical SK model with the convex correlation function G_r (see Remark 4.2.2). It is this convexity that allows for the proof of the upper bound (along the lines of Talagrand [186]) in (4.24). for all admissible correlators.

SKETCH OF THE PROOF. The proof of Theorem 4.3.2 exploits the observation (4.26) and combines it with the localization technique of Bovier & Klimovsky [44]. By means of the large deviations principle, this technique reduces the analysis of the full log-partition function (4.5) to the *local* one, where (4.26) approximately holds true everywhere. The price to pay for this reduction is the saddle point variational principle (4.24), which involves the Lagrange multipliers that enforce the localization. This gives the upper bound.

For the lower bound, on each spherical shell, we use the Aizenman-Simms-Starr scheme Aizenman *et al.* [2] following the beautiful method of Panchenko [166]. Next, using Ghirlanda-Guerra identities Ghirlanda & Guerra [98] and Panchenko [164], we show that the corresponding comparison structure converges to the Ruelle probability cascade (RPC) Panchenko [164] and Ruelle [174]. Finally, Aizenman-Sims-Starr scheme evaluated at the RPC yields the matching lower bound in (4.24).

4.4 ROTATIONALLY INVARIANT CONFIGURATION SPACE: THE FYODOROV-SOMMERS FORMULA

Parallel to the product state space (4.3), one can consider the *rotationally invariant state space*:

$$S_N := \{u \in \mathbb{R}^N : \|u\|_2 \leq L\sqrt{N}\}, \quad L > 0. \tag{4.28}$$

In this case, we assume that the a priori measure $\mu_N \in \mathcal{M}_{\text{finite}}(S_N)$ has the density

$$\frac{\mathrm{d}\mu_N}{\mathrm{d}\lambda_N}(u) := \exp\left(\sum_{i=1}^{N} f(u_i)\right), \quad u = (u_i)_{i=1}^N \in \mathbb{R}^N, \quad f\colon \mathbb{R} \to \mathbb{R} \tag{4.29}$$

with respect to the Lebesgue measure λ on $\mathbb{R}^N$. Let the function f be of the form $f(u) := h_1 u - h_2 u^2$, where $h_1 \in \mathbb{R}$ and $h_2 \in \mathbb{R}_+$ are given constants. Let us note that in case (4.28), $d = L^2$.

In the case of the rotationally invariant state space (4.28), one can obtain a more explicit representation for the Parisi functional (4.20), which does not require any regularization. Given $x \in \mathcal{X}(r)$, define $q_{\max} := q_{\max}(x) := \sup\{q \in [0;r] : x(q) < 1\}$. Consider the Crisanti-Sommers type functional (cf. Crisanti & Sommers [61, (A2.4)] and Fyodorov & Sommers [93, (47)])

$$\begin{aligned} \mathcal{CS}(\beta,r)[x] :=& \frac{1}{2}\left[\log(r-q_{\max}) + \int_0^{q_{\max}} \frac{\mathrm{d}q}{\int_q^r x(s)\mathrm{d}s} + h_1^2\int_0^r x(q)\mathrm{d}q - h_2 r\right] \\ &+ \frac{\beta^2}{2}\left(D'(2(r-q_{\max})) + \int_0^{q_{\max}} D'(2(r-q))x(q)\mathrm{d}q\right), \quad x \in \mathcal{X}(r). \end{aligned} \tag{4.30}$$

By reducing the case of the rotationally invariant state space to the product state space case via a large deviations argument (an idea exploited in Talagrand [184]), we arrive at the following.

Theorem 4.4.1 (Fyodorov-Sommers formula)**.** *In the case of the rotationally invariant state space (4.28), for all $\beta \in \mathbb{R}_+$, $h_1 \in \mathbb{R}$, $h_2 \in \mathbb{R}_+$, there exists unique $r^* \in [0;d]$ and unique $x^* \in \mathcal{X}(r)$ such that*

$$p(\beta) = \max_{r\in[0;d]} \min_{x\in\mathcal{X}(r)} \mathcal{CS}(\beta,r)[x] = \mathcal{CS}(\beta,r^*)[x^*], \quad \textit{almost surely.} \tag{4.31}$$

Remark 4.4.1. *The Crisanti-Sommers type functional* (4.30) *corresponds to the a priori distribution* (4.29)*, which represents the linear combination of linear and quadratic external fields. Formula [93, (47)] was derived under the assumption of the quadratic external field, whereas formula [61, (A2.4)] was obtained for the spherical SK model with the linear external field.*

4.5 RELATED RESEARCH

The model on rotationally invariant configuration space was studied in detail using physics heuristics (replica method + hierarchical replica symmetry breaking Ansatz) by Fyodorov & Le Doussal [91], Fyodorov & Bouchaud [92], and Fyodorov & Sommers [93] (see also older related physics literature, e.g., Mézard & Parisi [150]).

4.6 OUTLOOK

Here are some open problems related to the model considered in this chapter:

- For the rotationally invariant configuration space:
  - What is the random geometry/topology of the landscape? Are there many local minima? Are they far apart? Are they similarly deep? Are they separated by high barriers?
  - Identify the fluctuations of the partition function in the model of this chapter.
  - Study the phase diagram of the model at complex temperatures. Does it have features similar to the CREM, see Section 2.13?
- In the spirit of the equations of Thouless, Anderson & Palmer [187], derive a set of fixed–point equations for computing the expected (w.r.t. the Gibbs measure) values of the coordinates of the model. Some related mathematics literature is Belius & Kistler [23], Chen & Panchenko [53], and Chen *et al.* [54].

Part II

INFORMATION-BASED MODELS

5 SPATIAL CANNINGS MODEL IN RANDOM ENVIRONMENT

We introduce and study a system of hierarchically interacting measure-valued random processes that arises as the continuum limit of a large population of individuals carrying different types. Individuals live in colonies labelled by the hierarchical group of order N, and are subject to *migration* and *resampling* on all hierarchical scales simultaneously. The resampling mechanism is such that a random positive fraction of the population in a block of colonies inherits the type of a random single individual in that block, which is why we refer to our system as the hierarchical Cannings process. Before resampling in a block takes place, all individuals in that block are relocated uniformly, which we call *reshuffling*. The evolution of the system seen backwards in time leads to a dual process of coalescing random walks (representing the lineages) in random environment. The space-time scaling behaviour of the dual determines that of the system forward in time.

We study a version of the hierarchical Cannings process in *random environment*, namely, the resampling measures controlling the change of type of individuals in different blocks are chosen randomly with a given mean and are kept fixed in time, i.e., we work in the *quenched* setting. We give a necessary and sufficient condition under which a *multi-type* equilibrium is approached (= coexistence) as opposed to a *mono-type* equilibrium (= clustering). Moreover, in the hierarchical mean-field limit $N \to \infty$, with the help of a *renormalization analysis* we obtain a full picture of the space-time scaling behaviour of block averages on all hierarchical scales simultaneously. We show that the k-block averages are distributed as the superposition of a Fleming-Viot diffusion with a deterministic volatility constant d_k and a Cannings process with a random jump rate, both depending on k. In the random environment d_k turns out to be smaller than in the homogeneous environment of the same mean.

This chapter is based on publications 1a., 3a., 7a.

5.1 MOTIVATION AND GOAL

Two models play a central role in the world of stochastic multi-type population dynamics:

(1) The Moran model and its limit for large populations, the Fleming-Viot measure-valued *diffusion*.

(2) The Cannings model and its limit for large populations, the Cannings measure-valued *jump process* (also called the generalized Fleming-Viot process).

The Cannings model accounts for situations in which *resampling* is such that a random positive fraction of the population in the next generation inherits the type of a random single individual in the current generation, even in the infinite population limit (see Cannings [48], [49]). In order to describe a setting where this effect has a geographical structure, i.e., where *migration* of individuals is allowed as well, different models have been proposed in Limic and Sturm [142], Blath, Etheridge and Meredith [34], Barton, Etheridge and Véber [21], Berestycki, Etheridge and Véber [26]. The behaviour of these models has been studied in detail and its dependence on the geographic space is fairly well understood.

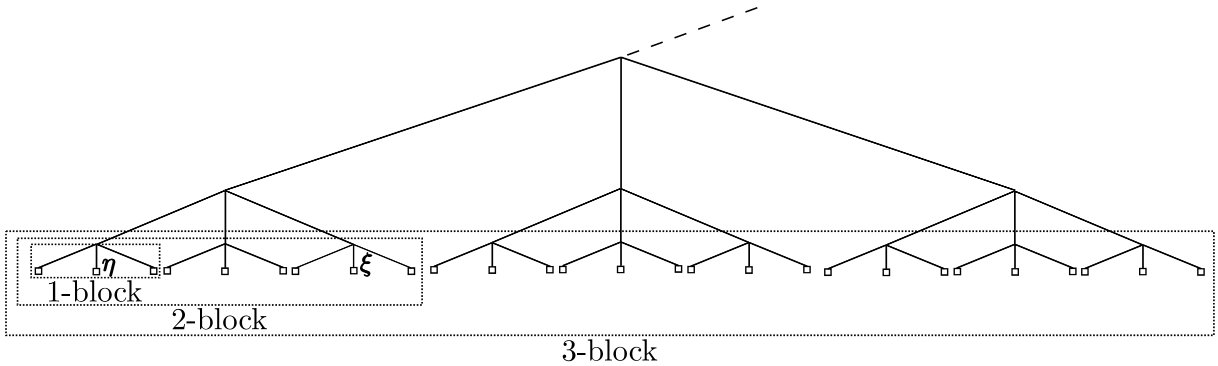


Figure 5.1: Hierarchical group

The type space is typically chosen to be a compact Polish space E. In this chapter, we focus on the case where the geographic space is the hierarchical group Ω_N of order N (see Figure 5.1), since this allows us to carry out a full *renormalization analysis*. In the *hierarchical mean-field limit* $N \to \infty$, the migration can be chosen in such a way that it approximates migration on the geographic space $\mathbb{Z}^2$, a possibility that was exploited by Sawyer and Felsenstein [179] (see also Dawson *et al.* [68]).

In this chapter, we focus on the spatial Cannings model the reproduction mechanism is controlled by *catastrophic events* on a small time scale, for which it is appropriate to assume that the rate of occurrence has a spatially inhomogeneous structure. This leads us to consider spatial Cannings models with block resampling in *random environment*, i.e., both the form and the overall rate of the block resampling mechanism depend on the geographic location.

Remark 5.1.1. We *only* work with continuum models. However, we motivate these models by viewing them as the large-population limit of individual-based models.

Our *goal* is three-fold:

(1) *Construction* of the hierarchical Cannings process in random environment via a well-posed martingale problem and derivation of a *duality relation* with a hierarchical spatial coalescent in random environment.

(2) Analysis of the longtime behaviour, in particular, the *dichotomy* between a multi-type equilibrium and a mono-type equilibrium.

(3) Scaling analysis of a collection of *renormalised processes* obtained by looking at the evolution of blocks averages on successive space-time scales in the hierarchical mean-field limit and the consequences for universality classes of the mono-type cluster formation.

5.2 MIGRATION IN A HIERARCHICAL GEOGRAPHY

Sawyer & Felsenstein [179] suggested a model with migration rates that do not depend on the Euclidean distance between the colonies but rather on the clustering distance (e.g., village $\rightsquigarrow$ valley $\rightsquigarrow$ province $\rightsquigarrow$ state $\rightsquigarrow$ country $\rightsquigarrow$ continent).

Consider the following geographical space:

- (Countable abelian) *Hierarchical group* (which can also be seen as a regular tree): $\Omega_N = \left\{\eta = (\eta^l)_{l\in\mathbb{N}_0} \in \{0,1,\dots,N-1\}^{\mathbb{N}_0} \colon \sum_{l\in\mathbb{N}_0} \eta^l < \infty\right\}$.
- Here $N \in \mathbb{N}$ is a parameter.
- We interpret $\eta \in \Omega_N$ as the *address of a colony* in the geographical space, see Figure 5.2.

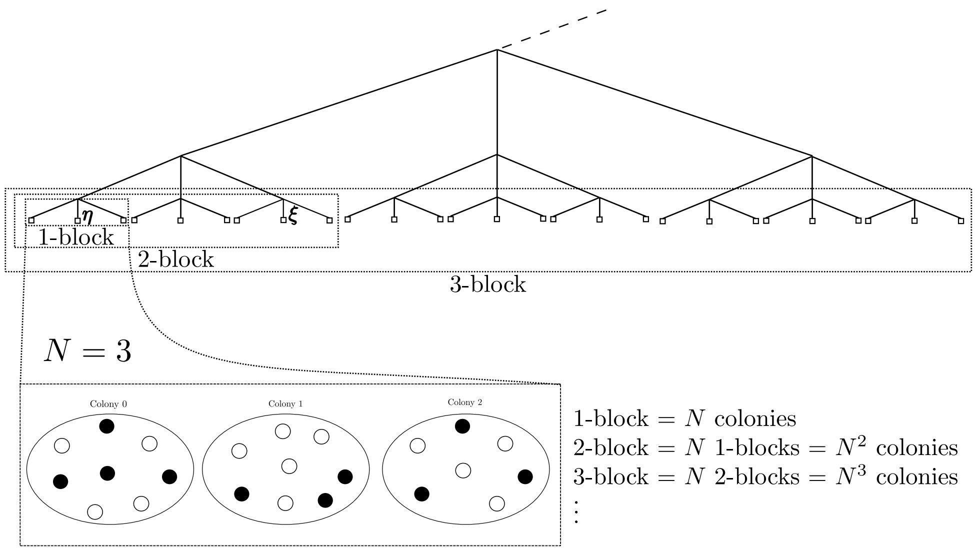


Figure 5.2: A hierarchical geographical space with colonies of multi-type individuals

MIGRATION ON THE HIERARCHICAL SPACE. Dawson *et al.* [68] introduced and studied in detail the *hierarchical random walk* (HRW). This is a random walk on the hierarchical group Ω_N with the following ingredients:

- *Migration rates:* $\underline{c} := (c_k)_{k \in \mathbb{N}_0} \in (0, N)^{\mathbb{N}_0}$.
- Each individual at $\eta \in \Omega_N$ jumps uniformly in a k-block around η at rate c_{k-1}/N^{k-1}.

5.3 Λ-CANNINGS MODEL

REPRODUCTION WITHIN A COLONY: THE DISCRETE CANNINGS MODEL. The *Cannings model* in discrete time was introduced by Cannings [48, 49] and is based on the following ingredients:

- $M \in \mathbb{N}$ the population size.
- *Exchangeable* collection of r.v. $\{\nu_i^{(M)} \in [0:M] : i \in [1:M]\}$ representing the numbers of children for each individual currently alive, which
- satisfies the constraint $\sum_{i=1}^{M} \nu_i^{(M)} = M$ of constant population size.

Λ-CANNINGS MODEL. In a continuous time, continuous mass limit, discrete Cannings models can be rescaled to the Λ-*Cannings model*. This large universality class appearing in the limit of $M \to \infty$ was identified by Sagitov [176], Möhle & Sagitov [152]:

In the multi-type situation, it is convenient to encode the state of the colony via the empirical distribution of types. For $M \to \infty$, we study the evolution of the *empirical distribution of types*: $X(t) := \frac{1}{M} \sum_{i=1}^{M} \delta_{T(i,t)} \in \mathcal{M}_1(E)$ in a colony. The evolution is Markovian and is driven by the *Poisson point process* on $\mathbb{R}_+ \times [0,1]$ with $\mathrm{d}t \otimes \Lambda(\mathrm{d}r)/r^2$, where $\Lambda \in \mathcal{M}_{\mathrm{finite}}([0,1])$, $\Lambda(\{0\}) = 0$. At each jump, the population is resampled: the individuals are marked for resampling by the *Bernoulli experiment* $(r\delta_1 + (1-r)\delta_0)^{\otimes M}$ and the ones marked all get the same type of a randomly chosen parent individual, see Figure 5.3

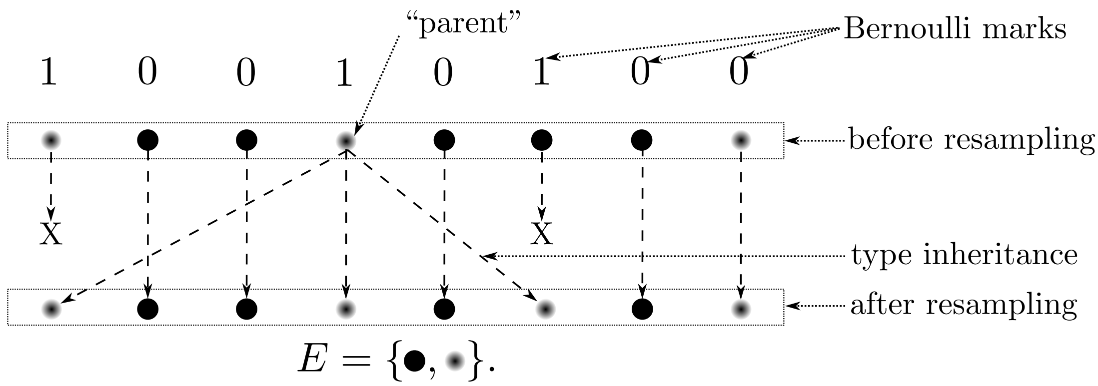


Figure 5.3: Resampling

5.4 INHOMOGENEOUS CATASTROPHES

Now we introduce the *catastrophes* affecting the whole blocks:

- They can model, e.g., droughts, floods, forest fires, epidemics, meteorite impacts, etc.

- They are driven by the Poisson point process on $\mathbb{R}_+ \times [0,1]$ with $\mathtt{d}t \otimes N^{-2k}\Lambda_k(\mathtt{d}r)/r^2$, where $\Lambda_k \in \mathcal{M}_1([0,1])$, $\Lambda(\{0\}) = 0$.
- Non-local resampling-reshuffling:
  - *Reshuffle* all individuals simultaneously in the k-block by assigning to each a new uniformly chosen location in the block.
  - *Resample* the individuals in k-block using the Cannings mechanism Λ_k as if they were at the same location.

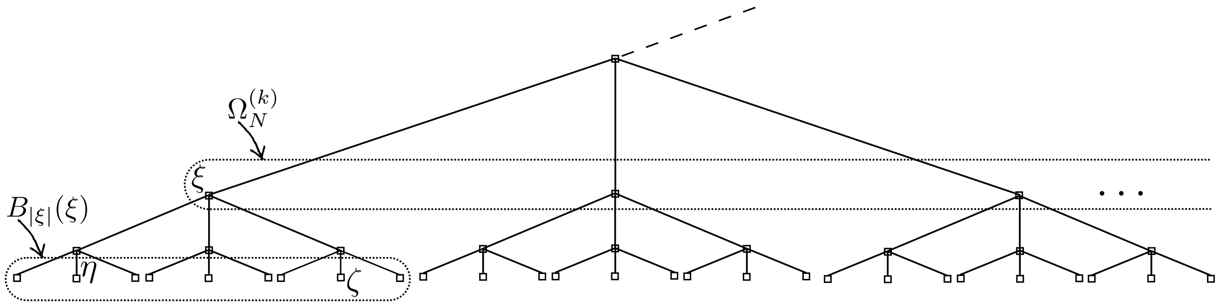


Figure 5.4: $\Omega_N^{\mathbb{T}}$ with $N = 3$, $\xi \in \Omega_N^{(k)} \subset \Omega_N^{\mathbb{T}}$, $|\xi| = k = 2$, $\eta, \zeta \in B_{|\xi|}(\xi)$. The elements of $\Omega_N^{\mathbb{T}}$ are the vertices of the tree (indicated by □'s).

SPATIALLY INHOMOGENEOUS ENVIRONMENT. Now we introduce the *spatially inhomogeneous mechanism of catastrophes*. Similarly to the Λ-resampling, the catastrophes are driven by the random measures:

$$\underline{\Lambda}(\omega) = \{\Lambda^{\xi}(\omega) \in \mathcal{M}_f([0,1]) \colon \ \xi \in \Omega_N^{\mathbb{T}}\} \tag{5.1}$$

where $\Omega_N^{\mathbb{T}}$ is the full tree (i.e., not only leafs but all other nodes of the tree), see Figure 5.4.

- $\xi \in \Omega_N^{\mathbb{T}}$ is the *address of a* $|\xi|$*-block*.
- ω is the *random environment*.
- Structural assumption: $\Lambda^{\xi}(\omega) = \lambda_{|\xi|}\chi^{\xi}(\omega)$, where:
  - $\chi^{\xi}(\omega) \in \mathcal{M}_f([0,1])$ is random stationary and λ_k is deterministic.

SUMMARY OF THE HIERARCHICAL CANNINGS PROCESS IN RANDOM ENVIRONMENT. We have informally described the dynamics of the hierarchically interacting $(\underline{c}, \underline{\Lambda})$-Cannings process in random environment, see Figure 5.2.

$$X^{(\Omega_N)} = \{X_{\eta}^{(\Omega_N)}(\omega; t) \in \mathcal{M}_1(E)\}_{t \in \mathbb{R}_+, \eta \in \Omega_N}. \tag{5.2}$$

The dynamics features a *competition between:*

- *Migration* $\underline{c} = (c_k)_{k \in \mathbb{Z}_+}$ (spatial movement) **vs.** *Resampling* + *Catastrophes* in *random environment* $\underline{\Lambda} = (\Lambda_k(\omega))_{k \in \mathbb{Z}_+}$ (reproduction under constrained resources).

- The competition happens on the hierarchy of *slow* and *fast time scales* on which the blocks evolve.

Remark 5.4.1. *The dynamics has the following features:*

- *Non-diffusive behaviour because of the PPP-driven jumps.*
- *Strongly correlated global updates because of the non-local reshuffling-resampling.*
- *Spatially inhomogeneous jump rates because of the random environment.*

TECHNICAL ASSUMPTIONS. In what follows, we impose some technical assumptions on the resampling mechanism:

$$\Lambda_0(\{0\}) = 0, \qquad \int_{(0,1]} \frac{\Lambda_0(\mathrm{d}r)}{r} = \infty, \tag{5.3}$$

and

$$\Lambda_k(\{0\}) = 0, \qquad \int_{(0,1]} \frac{\Lambda_k(\mathrm{d}r)}{r^2} < \infty. \qquad k \in \mathbb{N}. \tag{5.4}$$

5.5 LONG-RUN BEHAVIOR

Question 5.5.1. *Assume that we start the spatial Cannings model in random environment in a biodiverse configuration.*

- *Does the process converge to equilibrium in the long run, i.e.*

$$\mathcal{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \textbf{?} \tag{5.5}$$

- *Is there a biodiversity in the long run?*

The answer to the first part of the question is as follows.

Theorem 5.5.1 (Equilibrium). *Fix $N \in \mathbb{N}\backslash\{1\}$. Suppose that, under the law $\mathbb{P}$, the law of the initial state $X^{(\Omega_N)}(\omega;0)$ is* ***stationary and ergodic*** *under translations in $\Omega_N^{\mathbb{T}}$, with mean single-coordinate measure $\theta = \mathbb{E}[X_0^{(\Omega_N)}(\omega;0)] \in \mathcal{P}(E)$. Then, there exists an equilibrium measure $\nu_\theta^N(\omega) \in \mathcal{P}(\mathcal{P}(E)^{\Omega_N})$:*

$$\lim_{t\to\infty} \mathcal{L}\left[X^{(\Omega_N)}(\omega;t)\right] = \nu_\theta^N(\omega), \quad \mathbb{P}\text{-a.s. } \omega \tag{5.6}$$

satisfying

$$\int_{\mathcal{P}(E)^{\Omega_N}} x_0\, \nu_\theta^N(\omega;\mathrm{d}x) = \theta. \tag{5.7}$$

Moreover, under the law $\mathbb{P}$, $\nu_\theta^N(\omega)$ is stationary and ergodic under translations in $\Omega_N^{\mathbb{T}}$.

CLUSTERING VS. COEXISTENCE. To answer the second part of Question 5.5.1, we consider the following two scenarios for the equilibrium $\nu_\theta^N(\omega)$:

- **Coexistence given the environment ω:**

$$\sup_{\psi \in C_b(E)} \int_{\mathcal{P}(E)^{\Omega_N}} \nu_\theta^N(\omega)(\mathrm{d}x) \left[\int_E \psi^2(u) x_0(\mathrm{d}u) - \left(\int_E \psi(u) x_0(\mathrm{d}u) \right)^2 \right] > 0. \tag{5.8}$$

  In words: The variance of the type distribution is positive.

- **Clustering given the environment ω:**

$$\nu_\theta^N(\omega) = \int_E \delta_{(\delta_u)^{\Omega_N}} \theta(\mathrm{d}u). \tag{5.9}$$

  In words: The system grows mono-type clusters that cover the whole Ω_N.

We obtain the following dichotomy between clustering and coexistence for $N < \infty$.

Theorem 5.5.2 (Dichotomy for finite N). *Fix $N \in \mathbb{N} \backslash \{1\}$ and assume that $\rho^\xi(\omega) := \chi^\xi([0,1], \omega)$ satisfies*

$$\mathbb{E}[\rho^\xi(\omega)] = 1, \qquad \exists\, \delta > 0 \colon\ \delta \le \rho^\xi(\omega) \le \delta^{-1}\ \forall\, \xi \in \Omega_N \text{ for } \mathbb{P}\text{-a.e. } \omega. \tag{5.10}$$

(a) *Let $\mathcal{C}_N := \{\omega \colon$ coexistence given ω occurs$\}$. Then, $\mathbb{P}(\mathcal{C}_N) \in \{0,1\}$.*

(b) $\mathbb{P}(\mathcal{C}_N) = 1$ *iff*

$$\sum_{k \in \mathbb{N}_0} \frac{1}{c_k + N^{-1}\lambda_{k+1}} \sum_{l=0}^{k} \lambda_l < \infty. \tag{5.11}$$

Remark 5.5.1. *Criterion (5.11) implies:*

- *Sublte inteplay between the migration c and resampling/catastrophe λ parameters with respect to their influence on clustering/coexistence.*

- *If $\lambda_l = 0$, $\forall l \ge 1$ (i.e., no catastrophes), then the criterion reduces to the recurrence condition for the migration.*

IDEA OF PROOF. Study the *lineages* of particles (backwards in time). It turns out that the backwards in time dynamics is in a sense simpler:

- **Duality**: lineages evolve according to a *spatial coalescent with non-local coalescence in random environment*.

This idea is formalized in the next paragraph.

DUALITY WITH A SPATIAL COALESCENT WITH NON-LOCAL COALESCENCE IN RANDOM ENVIRONMENT. The idea of duality is to relate $X = \{X_t\}_{t\in\mathbb{R}_+}$ with a *simpler stochastic process*. A possible formulation is as follows. Find H and $Y = \{Y_t\}_{t\in\mathbb{R}_+}$:

$$\mathbb{E}_{X_0}[H(X_t, Y_0)] = \mathbb{E}_{Y_0}[H(X_0, Y_t)], \text{ for all } (X_0, Y_0), \quad t \in \mathbb{R}_+. \tag{5.12}$$

In our case, a useful *dual* is provided by the so-called **spatial coalescent with non-local coalescence in random environment** $\{Y_t\}_{t\in\mathbb{R}_+}$. The coalescent can be seen as the *backwards-in-time* dynamics of the *coalescing lineages* in the Cannings process.

- The initial configuration consists of infinitely many singleton *families*.
- Families move around according to the HRW.
- The coalescence events are driven by the *PPP* with intensity
$$\mathrm{d}t \otimes \mathrm{d}\eta \otimes \left(N^{-2k}\mathrm{d}k \left[\Lambda_k(\mathrm{d}r)\big(r\delta_1 + (1-r)\delta_0\big)^{\otimes\mathbb{N}}\right](\mathrm{d}\omega)\right). \tag{5.13}$$
- At a coalescence event, $k \geq 2$ families in $B_k(\eta)$ coalesce. Immediately afterwards, all families in B_k are reshuffled (= randomly and simultaneously relocated in within B_k).

BIODIVERSITY DICHOTOMY: CLUSTERING VS. COEXISTENCE. *Dichotomy* can be understood from the backwards in time viewpoint:

- If there is a single family in the long run, then there is no *biodiversity* (clustering).
- If there is more then one family in the long run, there is *coexistence*.

Exchangeability combined with duality implies that it is enough to consider *two coalescing random walks* $(Z_t^1(\omega), Z_t^2(\omega))_{t\geq 0}$ on Ω_N with migration coefficients $(c_k + \lambda_{k+1}N^{-(k+1)})_{k\in\mathbb{N}_0}$ in random environment and coalescence at rates $(\lambda_k = \Lambda_k([0,1]))_{k\in\mathbb{N}_0}$. The time-$t$ accumulated hazard for coalescence of this pair reads:

$$H_N(\omega;t) = \sum_{k\in\mathbb{N}_0} N^{-k} \sum_{\substack{\eta,\eta'\in\Omega_N \\ d_{\Omega_N}(\eta,\eta')\leq k}} \lambda^{\mathrm{MC}_k(\eta)}(\omega) \int_0^t 1_{\{Y_s(\omega)=\eta, Y_s'(\omega)=\eta'\}}\,\mathrm{d}s, \tag{5.14}$$

We can show the following:

Lemma 5.5.1.

- $\lim_{t\to\infty} H_N(t;\omega) = \infty$ *a.s.* $\rightsquigarrow$ *no biodiversity (clustering).*
- $\lim_{t\to\infty} H_N(t;\omega) < \infty$ *a.s* $\rightsquigarrow$ *coexistence.*

5.6 LARGE SPACE-TIME SCALE ANALYSIS

Question 5.6.1 (Coarse-grained dynamics)**.** *Where does the law of the k-block average converges to? I.e.,*

$$\mathcal{L}\left[k\text{-block average}(t\cdot N^k;\omega)\right] \underset{N\to+\infty}{\Longrightarrow} \textbf{?}$$

Is there a universal limiting law?

In order to answer this question, we perform the large space-time scale analysis.

LARGE SPACE-TIME SCALE ANALYSIS: $N \to \infty$.

- We *"separate" slow and fast time scales.* This is guaranteed in our model in the limit $N \to \infty$.
- This way, we can analyse the system scale by scale: starting from small scales and working our way towards larger scales. This can be seen as an orbit of the renormalization group.
- Specifically, consider the *macroscopic observables:* block averages of order $k \in \mathbb{Z}_+$

$$Y^{(N)}_{\eta,k}(tN^k;\omega) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X^{(\Omega_N)}_{\zeta}(tN^k;\omega), \qquad \eta \in \Omega_N,\ k \in \mathbb{Z}_+ \tag{5.15}$$

- The Cannings model on a fully connected geographical space (mean field or single scale situation), see Figure 5.5, decorellates (*propagation of chaos*) in the long run and each colony follows a *McKean-Vlasov process.*

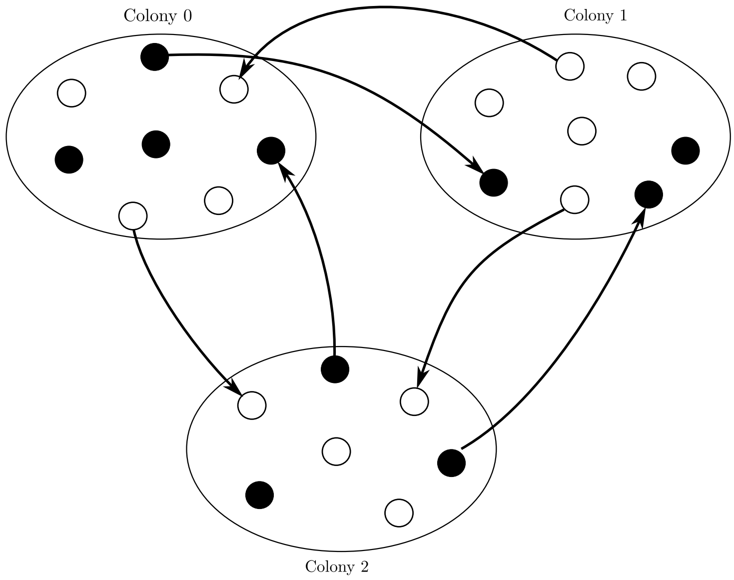


Figure 5.5: Mean-field migration of individuals between the $N = 3$ colonies (rate c/N random walk on the full graph)

MCKEAN-VLASOV LIMITING DYNAMICS. To describe the limiting measure-valued McKean-Vlasov dynamics, we need an *algebra of test functions*: $\mathcal{B} \subseteq C_b(\mathcal{M}_1(E), \mathbb{R})$ with $G \in \mathcal{B}$:

$$G(x) = \int_{E^n} x^{\otimes n}(\mathrm{d}u)\, \varphi(u), \qquad x \in \mathcal{M}_1(E),\ n \in \mathbb{N},\ \varphi \in C_b(E^n, \mathbb{R}). \tag{5.16}$$

The *Generator* turns out to be

$$\begin{aligned}
(L^{c,d,\Lambda}_{\theta} G)(x) &= c \int_E (\theta - x)\,(\mathrm{d}a)\, \frac{\partial G(x)}{\partial x}[\delta_a] \leftarrow \textbf{[drift]} \\
&+ d \int_E \int_E Q_x(\mathrm{d}u, \mathrm{d}v)\, \frac{\partial^2 G(x)}{\partial x^2}[\delta_u, \delta_v] \leftarrow \textbf{[Fleming-Viot diffision]} \\
&\quad + \int_{[0,1]} \Lambda^*(\mathrm{d}r) \int_E x(\mathrm{d}a)\, \big[G\big((1-r)x + r\delta_a\big) - G(x)\big] \leftarrow \textbf{[jumps]}, \quad G \in \mathcal{B},
\end{aligned} \tag{5.17}$$

where

$$Q_x(\mathrm{d}u, \mathrm{d}v) = x(\mathrm{d}u)\,\delta_u(\mathrm{d}v) - x(\mathrm{d}u)\,x(\mathrm{d}v). \tag{5.18}$$

A Markov process with the limiting generator we call the C^Λ-*processes with immigration-emigration* and denote it by

$$Z_\theta^{c,d,\Lambda} = \left(Z_\theta^{c,d,\Lambda}(t)\right)_{t\geq 0}, \quad Z_\theta^{c,d,\Lambda}(0) = \theta. \tag{5.19}$$

5.7 RENORMALIZATION AND MULTI-SCALE ANALYSIS

Theorem 5.7.1 (Behaviour of the macroscopic observables)**.** *Suppose that for each N the random field $X^{(\Omega_N)}(\omega;0)$ is the restriction to Ω_N of a random field $X(\omega)$ indexed by $\Omega_\infty = \bigoplus_{\mathbb{N}} \mathbb{N}$ that is i.i.d. with single-component mean $\theta \in \mathcal{P}(E)$. Then, for $\mathbb{P}$-a.e. ω and every $k \in \mathbb{N}$ and $\eta \in \Omega_\infty$,*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(Y_{\eta,k}^{(\Omega_N)}(\omega;tN^k)\right)_{t\geq 0}\right] = \mathcal{L}\left[\left(Z_\theta^{c_k,d_k,\Lambda^{\mathrm{MC}_k(\eta)}(\omega)}(t)\right)_{t\geq 0}\right], \tag{5.20}$$

where

- **Volatility constants:** $\underline{d} = (d_k)_{k\in\mathbb{Z}_+}$, are given recursively by

$$d_0 = 0, \qquad d_{k+1} = \mathbb{E}_{\mathcal{L}_\rho}\left[\frac{c_k(\mu_k\rho + d_k)}{c_k + (\mu_k\rho + d_k)}\right], \quad k \in \mathbb{Z}_+, \tag{5.21}$$

  where $\mu_k = \lambda_k/2$ and $\mathcal{L}[\rho] = \mathcal{L}[\rho^0]$.

- **N.B.** The formula (5.21) is an average of a *random Möbius transformation*. For a detailed analysis of the iterations of (5.21), we refer to Greven *et al.* [101].

HEURISTIC DERIVATION OF THE FORMULA FOR VOLATILITIES. Formula (5.21) can heuristically be understood as follows:

- The space-time scales separate, as $N \to \infty$. Therefore, it is enough to consider the 1-block averages.
- Due to duality it is enough to focus on study the coalescing lineages.
- Total coalescence rate = volatility.
- At space-time scale Nt, **only pairs of lineages can possibly meet** (cf. Limic & Sturm [142] for a related setup).
- The lineages coalesce at the rate $\lambda^{(\eta,0)}(\omega) = \Lambda^{(\eta,0)}((0,1])(\omega)$, if they are in the same colony.
- Probability to migrate away before coalescence is $2c_0/(2c_0 + \lambda^{(\eta,0)}(\omega))$.
- Therefore, the average **total coalescence rate** equals

$$\mathbb{E}\left[\frac{2c_0\lambda_0\rho(\omega)}{2c_0 + \lambda_0\rho(\omega)}\right]. \tag{5.22}$$

RANDOM ENVIRONMENT FACILITATES BIODIVERSITY. The volatility d_k in the random environment can be sandwiched between:

- the volatility d_k^0 in the *zero environment* ($\mathcal{L}_\rho = \delta_0$, i.e., the system without resampling)
- the volatility d_k^1 in the *average environment* ($\mathcal{L}_\rho = \delta_1$, i.e., the system with average resampling).

Theorem 5.7.2 (Randomness lowers volatility). *Assume the environment is non-deterministic. If $d_0^0 = d_0 = d_0^1$, then*

$$d_k^0 < d_k < d_k^1, \quad k \in \mathbb{N}. \tag{5.23}$$

Proof. Jensen's inequality. □

MULTI-SCALE ANALYSIS. When viewed on multiple time scales, the block averages converge to a time inhomogeneous measure-valued Markov chain. The transition kernel of the Markov chain is given by the quasi-equilibrium with the initial condition being the current state of the chain.

Theorem 5.7.3 (Multi-scale behaviour). *Let $(t_N)_{N\in\mathbb{N}}$ be such that $\lim_{N\to\infty} t_N = \infty$ and $\lim_{N\to\infty} t_N/N = 0$. Then, for $\mathbb{P}$-a.e. ω, every $j \in \mathbb{N}$ and every $\eta \in \Omega_\infty$,*

$$\lim_{N\to\infty} \mathcal{L}\left[\left(Y_{\eta,k}^{(\Omega_N)}(\omega; t_N N^k)\right)_{k=-(j+1),-j,\dots,0}\right] = \mathcal{L}\left[\left(M_{\eta,k}^{(j)}(\omega)\right)_{k=-(j+1),-j,\dots,0}\right], \tag{5.24}$$

where $M_\eta^{(j)}(\omega) = (M_{\eta,k}^{(j)}(\omega))_{k=-(j+1),-j,\dots,0}$ is the time-inhomogeneous Markov chain with initial state

$$M_{\eta,-(j+1)}^{(j)}(\omega) := \theta, \tag{5.25}$$

and transition kernel from time $-(k+1)$ to $-k$ given by

$$K_{\eta,k}(\omega; \theta, \cdot) := \nu_\theta^{c_k, d_k, \Lambda^{\mathrm{MC}_k(\eta)}(\omega)}(\cdot). \tag{5.26}$$

5.8 RELATED RESEARCH

For an introduction to the theory of measure-valued stochastic population dynamics, we refer to, e.g., Dawson [67] and Etheridge [85].

RANDOM SYSTEMS IN ULTRAMETRIC SPACES. A recent review of dynamics on is Dawson & Gorostiza [64].

SPATIAL CANNINGS MODEL. Spatial Cannings models on Euclidean spaces were suggested in Barton *et al.* [21] and Etheridge [86] and further studied in [26].

MULTI-SCALE ANALYSIS OF STOCHASTIC PROCESSES ON HIERARCHICAL NETWORKS. Renormalization is a key method to analyze large space-time behavior and universality in interacting particle systems Kadanoff [125]. For a review in the context of interacting population models before 2005, we refer to Greven [100]. The closest in spirit work to this chapter is Dawson *et al.* [65], where a *hierarchically interacting Fleming-Viot process* have been studied using multi-scale methods. Recent monograph Dawson & Greven [69] focuses on the spatial Fleming-Viot model with selection and mutation.

Several interesting IPS and related stochastic processes have been analyzed using renormalization methods, e.g., the *contact process* by Athreya & Swart [14], *Kuramoto model* by Garlaschelli *et al.* [96], *percolation* by Dawson & Gorostiza [63] and Koval *et al.* [132] and *random walks on percolation clusters* by Dawson & Gorostiza [64].

5.9 SUMMARY AND OUTLOOK

We summarize results of this chapter on the hierarchical Cannings model in random environment as follows:

- There is the *clustering vs. local coexistence dichotomy* in the long-time behavior of the model. It is formulated in terms of the migration and resampling parameters $\underline{c}, \underline{\lambda}$ for *finite* N.
- We have identified the *space-time scaling behaviour* in the hierarchical mean-field limit $N \to \infty$.
- It turns out that the volatilities decrease in the inhomogeneous environment; the clusters grow slower.
- The fluctuations of the environment reduce clustering. This implies an *increased biodiversity*.

OPEN PROBLEMS:

- Recover the measure-valued process studied in this chapter as a limit of discrete IPS.
- Relax condition (5.4).
- Study a model without reshuffling, when a catastrophe occurs, cf. 5.4.
- Extend the model and the analysis to cover the case of the Ξ-Cannings models, in which several individuals can simultaneously make a macroscopic contribution in the next generation, see, e.g., Birkner *et al.* [29].
- Extend the model and the analysis to cover other genalogical forces like mutation, selection and recombination, see, e.g., Dawson & Greven [69].
- Extend the model and the analysis to "continuum" hierarchical geographical spaces, see, e.g., Evans & Fleischmann [87] and Greven *et al.* [102].
- Extend the model and analysis to cover inhomogeneous geographical spaces using models of complex networks, see, e.g., Aldous [3] and Allen *et al.* [6].

6 MODELS OF EVOLVING NETWORKS

What are the emerging global patterns in complex systems? How do they come about from the local behavior of the elements? In the last 20 years, complex networks became a key tool to model real-world complex systems in the sciences. Yet, the majority of networks evolve over time and this can have a substantial effect on the processes unfolding on them. How can one model large interacting particle systems on evolving networks?
Starting from some popular modeling assumptions like Markovianity, vertex exchangeability and subsampling consistency, we study models of evolving large weighted networks. In the large network limit, these lead to Markovian exchangeable arrays. This is a step towards understanding the limits of interacting particle systems on evolving networks.

This chapter is based on publication 8a.

FOCUS OF THIS CHAPTER. In this chapter, we extend the results of Crane [57, 58] on Markovian dyanmics of graph limits to the case of *weighted networks*. This is a step towards understanding the probabilistic limiting objects in large IPS on evolving networks.

BACKGROUND. In the last 20 years, *complex networks* became a key tool to model real-world complex systems in the sciences (e.g., Barabási [17], Dorogovtsev & Mendes [79], and Newman [158]). Yet, the majority of networks evolve over time and this can have a substantial effect on the processes unfolding on them (Holme & Saramäki [109], Porter & Gleeson [170, Chapter VII], Lambiotte & Masuda [135, Chapter 6]). Moreover, the influence can also go the other way around: processes happening on a network can affect the evolution of the network itself. This leads to what is called *coevolution* in adaptive networks (e.g., Gross & Sayama [103]) or more generally complex adaptive systems (e.g., Holland [108] and Levin [139]). Examples include epidemiological and ecological networks, neural networks, systems biology networks, social networks, financial markets, etc. In all these contexts, there is a great deal of uncertainty/volatility in the structure and dynamics of the complex system. Thus, it is natural to use *stochastic processes on random graphs* as the modeling framework (e.g., Aldous [3], Durrett [81], and van der Hofstad [188]). Nonetheless, the mathematical theory for *stochastic processes on evolving networks* is largely lacking.

CURRENT SCIENTIFIC DISCOURSE. A large class of complex systems can be modeled by a "population" of particles (or agents) living on the nodes of a network. As time progresses, these particles interact with each other by influencing each other's state (and possibly the underlying network itself) at given rates, where a state represents a particular property of the particle, e.g., an opinion or an infection status. Such *interacting particle systems* (IPS) is a key object of study in probability theory (see, e.g., Liggett [141]) and in the sciences (where they are known under various names, e.g., agent-based models [55, 84]). However, the majority of the mathematics research on IPS so far has concentrated on very regular deterministic underlying networks such as lattices or Euclidean spaces. Only in the last several years, the need to study IPS on complex networks has been advocated in the mathematics literature (see, e.g., Aldous [3]). Finally, very recently, stochastic processes on evolving networks have experienced a surge of interest in the sciences (see, e.g., Lambiotte & Masuda [135] and Porter & Gleeson [170]). Yet, the approaches there are mostly based on non-rigorous methods and the mathematical theory is still in its infancy.

In parallel, understanding the limiting behavior of the probabilistic models of complex networks and stochastic processes on them in the large-sample limit is critical to enabling *statistical modeling* (Kolaczyk & Csárdi [128]) and *inference algorithms* with good theoretical properties such as rigorous performance guarantees. This emerging area of research at the interface between *probability theory* and *mathematical statistics* is full of challenging open problems, see, e.g., Crane [56]. In particular Crane [56, Chapter 11], focuses on models of dynamic networks and reviews several papers of the author including Crane [57, 58].

6.1 EXCHANGEABLE RANDOM ARRAYS

We will consider arrays with values in an arbitrary Polish space S. This space will be endowed with its Borel σ-field $\mathcal{B}(S)$ and a compatible metric d_S, which we assume to be bounded by 1. We write $\mathcal{P}(S)$ for the set of all probability measures on $(S, \mathcal{B}(S))$ endowed with the topology of weak convergence, which is a Polish space as well.

A random S-valued array is a collection $Y = (Y_{ij})_{ij \in \mathbb{N}}$ of S-valued random variables on some probability space $(\Omega, \mathcal{A}, P)$. Otherwise said, Y is $\mathbb{S} := S^{\mathbb{N}^2}$-valued random variable. We endow $\mathbb{S}$ with the product topology and the compatible metric $d_{\mathbb{S}}(y, y') = \sum_{i,j \in \mathbb{N}} 2^{-i-j} d_S(y_{ij}, y'_{ij})$.

For an arbitrary set $A \subset \mathbb{N}$, we define $Y|_A = (Y_{ij})_{i,j \in A}$ to be the restriction of Y to the index set A. In particular, with $[n] := \{1, \ldots, n\}$, $Y|_{[n]}$ is its restriction to the first n coordinates, $Y|_{[n]} \in \mathbb{S}_n := S^{n^2}$.

Similarly, for every probability distribution ν on $\mathbb{S}$ (or on $\mathbb{S}_m$, $m \geq n$), we denote by $\nu|_{[n]}$ its image under the canonical restriction from $\mathbb{S}$ (or $\mathbb{S}_m$) to $\mathbb{S}_n$. It is a known fact that a sequence of probability measures $(\mu^k)_{k \geq 1}$ on $\mathbb{S}$ converges weakly to $\mu \in \mathcal{P}(\mathbb{S})$, iff all restrictions $\mu^k|_{[n]} \in \mathcal{P}(\mathbb{S}_n)$, converge weakly in $\mathcal{P}(\mathbb{S}_n)$, or equivalently $\mu^k(f) \to \mu(f)$, for every bounded continuous cylinder function f on $\mathbb{S}$.

Let Σ be the set of all permutations of integers, that is the set of all bijections of $\mathbb{N}$ which fix all but finitely many values; Σ_n denotes the set of all permutations of $[n]$. For an array Y and $\boldsymbol{\pi} = (\pi_1, \pi_2) \in \Sigma^2$, we define a new array $Y^{\boldsymbol{\pi}}$ by $Y^{\boldsymbol{\pi}}_{ij} = Y_{\pi_1(i)\pi_2(j)}$. For $\pi \in \Sigma$, we also define Y^π by $Y^\pi_{ij} = Y_{\pi(i)\pi(j)}$. An array Y is called *exchangeable* if

$$Y \overset{\text{law}}{=} Y^{\boldsymbol{\pi}}, \qquad \text{for every } \boldsymbol{\pi} \in \Sigma^2. \tag{6.1}$$

An array Y is called *weakly exchangeable*[1] if it is symmetric (i.e., $Y_{ij} = Y_{ji}$) and

$$Y \overset{\text{law}}{=} Y^\pi, \qquad \text{for every } \pi \in \Sigma. \tag{6.2}$$

The key result of the theory of random arrays is their characterisation due to Aldous [4] and Hoover [110] which can be viewed as a "two-dimensional version" of de Finetti's theorem.

Theorem 6.1.1. (a) *If $(Y_{ij})_{i,j\in\mathbb{N}}$ is an S-valued exchangeable array, then there exists a measurable function $f\colon [0,1]^4 \to S$ such that $Y \overset{\text{law}}{=} Y^\star$, where*

$$Y^\star_{ij} = f(U, U_i, U'_j, U_{ij}), \tag{6.3}$$

and U, $(U_i)_{i\in\mathbb{N}}$, $(U'_i)_{i\in\mathbb{N}}$, and $(U_{ij})_{i,j\in\mathbb{N}}$ are independent collections of Uniform$([0,1])$ *i.i.d. random variables.*

(b) *If $(Y_{ij})_{i,j\in\mathbb{N}}$ is an S-valued weakly exchangeable array, then the analogous statement holds with a function $f\colon [0,1]^4 \to S$ satisfying $f(\cdot, x, y, \cdot) = f(\cdot, y, x, \cdot)$, and with*

$$Y^\star_{ij} = Y^\star_{ji} = f(U, U_i, U_j, U_{ij}), \qquad i \geq j. \tag{6.4}$$

The representing function f of the Aldous-Hoover theorem is not uniquely determined. E.g., in the case (a), if two functions f and f' satisfy $f'(a,b,c,d) = f(T_1(a), T_2(b), T_3(c), T_4(d))$ for some measure preserving transformations $T_1, \dots, T_4$ of $[0,1]$, then the corresponding exchangeable arrays have the same distribution.

A (weakly) exchangeable array is called *dissociated* if

$$(Y_{ij} : i,j \leq n) \text{ is independent of } (Y_{ij} : i,j > n), \text{ for each } n. \tag{6.5}$$

It is obvious that if the function f in the representation of Theorem 6.1.1 does not depend on the first coordinate, then Y is dissociated. Converse statement hold as well, see Corollary 14.13 in [5].

Dissociated arrays play a similar role as i.i.d. sequences do in the theory of exchangeable sequences: Every (weakly) exchangeable array is a mixture of (weakly) exchangeable dissociated arrays. To state this more formally, we need more definitions.

A set $A \in \mathcal{B}(\mathbb{S})$ is called *exchangeable* if $A = A^{\boldsymbol{\pi}}$ for every $\boldsymbol{\pi} \in \Sigma^2$, where $A^{\boldsymbol{\pi}} = \{y^{\boldsymbol{\pi}} : y \in A\}$ and $y^{\boldsymbol{\pi}}_{ij} = y_{\pi_1(i)\pi_2(j)}$. The collection $\mathcal{E}_S \subset \mathcal{B}(\mathbb{S})$ of all exchangeable sets is called the *exchangeable σ-field*. For an exchangeable array Y, we define $\mathcal{E}_Y = \{Y^{-1}(A) : A \in \mathcal{E}_S\}$. We use $\mathcal{D}_S \subset \mathcal{P}(\mathbb{S})$ to denote the set of all distributions of dissociated exchangeable arrays, which is a closed subset of $\mathcal{P}(\mathbb{S})$. We write $\tilde{\mathcal{D}}_S$ for the set of all distributions of dissociated weakly exchangeable arrays.

The following proposition follows from [5, Proposition 14.8 and Theorem 12.10].

Proposition 6.1.1. (a) *A (weakly) exchangeable array Y is dissociated iff $P(A) \in \{0,1\}$ for every $A \in \mathcal{E}_Y$, that is its exchangeable σ-field is P-trivial.*

(b) *Let Y be a (weakly) exchangeable array and α its regular conditional distribution given $\mathcal{E}_Y$. Then, $\alpha(\omega) \in \mathcal{D}_S$ (resp. $\alpha(\omega) \in \tilde{\mathcal{D}}_S$) for P-a.e. ω. Moreover, the distribution μ_Y of Y can be written as*

$$\mu_Y(\cdot) = \int_{\mathcal{D}_S} \nu(\cdot)\Lambda_Y(\delta\nu) \tag{6.6}$$

for a uniquely determined probability measure Λ_Y on $\mathcal{D}_S$ (resp. $\tilde{\mathcal{D}}_S$).

1 The terminology is slightly misleading: due to the symmetry requirement, the weak exchangeability is not weaker than the exchangeability

An important feature of exchangeable arrays is that regular conditional distribution α of Y given $\mathcal{E}_Y$ can, a.s., be recovered from a realisation of Y by the following procedure. For $m \geq n$, and $y \in \mathbb{S}$, let $t_m^{y,n} \in \mathcal{P}(\mathbb{S}_n)$ be defined by

$$t_m^{y,n} = \frac{1}{((m)_n)^2} \sum_{\psi_1,\psi_2} \delta_{(y_{\psi_1(i),\psi_2(j)})_{i,j\in[n]}}, \tag{6.7}$$

where the sum runs over all injections $\psi_1, \psi_2 \colon [n] \to [m]$ and

$$(m)_n = m(m-1)\dots(m-n+1). \tag{6.8}$$

Measure $t_m^{y,n}$ can be viewed as the empirical distribution of $n \times n$ sub-arrays in the array $y|_{[m]}$. We further define

$$t^{y,n} = \lim_{m\to\infty} t_m^{y,n} \tag{6.9}$$

whenever this limit exists in the weak sense, and set $|y| = (t^{y,n})_{n\geq 1}$ whenever all $t^{y,n}$, $n \geq 1$, exist.

It follows from the construction that the probability measures $t_m^{y,n}$, $n = 1, \dots, m$, are consistent in the sense that $t_m^{y,n}|_{[n-1]} = t_m^{y,n-1}$ for every $2 \leq n \leq m$. This consistence transfers to the limit, that is

$$t^{y,n}|_{[n-1]} = t^{y,n-1}, \qquad \text{for every } n \geq 2. \tag{6.10}$$

Therefore, in view of Kolmogorov's extension theorem, $|y|$, when it exists, can be viewed as an element of $\mathcal{P}(\mathbb{S})$.

In the weakly exchangeable case, we set $\tilde{t}_m^{y,n}$ by

$$\tilde{t}_m^{y,n} = \frac{1}{(m)_n} \sum_{\psi} \delta_{(y_{\psi(i),\psi(j)})_{i,j\in[n]}}, \tag{6.11}$$

where the sum runs over all injections π from $[n]$ to $[m]$. We then define $\tilde{t}^{y,n}$ and $|y| = (\tilde{t}^{y,n})_{n\geq 1}$ analogously as in the exchangeable case.

The next proposition establishes the connection between $|Y|$ and its regular conditional distribution α.

Proposition 6.1.2. (a) *Let Y be a (weakly) exchangeable array and α its regular conditional distribution given $\mathcal{E}_Y$. Then, for P-a.e. ω, $|Y(\omega)|$ exists and equals to $\alpha(\omega)$. In particular, $|Y(\omega)| \in \mathcal{D}_S$ (resp. $|Y(\omega)| \in \tilde{\mathcal{D}}_S$), P-a.s.*

(b) *If Y is dissociated, then $|Y|$ exists a.s. and coincides with the distribution of Y.*

Remark 6.1.1. *For the rest of the chapter, it will be suitable to extend the definition of $|y|$ to all possible $y \in \mathbb{S}$. For those $y \in \mathbb{S}$ for which some of the limits $t^{y,n}$ do not exist, we define $|y| = \partial$, where $\partial \notin \mathcal{P}(\mathbb{S})$ is an arbitrary symbol. In addition, for y such that $|y|$ exists but is not in $\mathcal{D}_S$, we set $|y| = \partial$ as well. By Proposition 6.1.2, we can then view $|y|$ as a map from $\mathbb{S}$ to $\mathcal{D}_S^\star := \mathcal{D}_S \cup \{\partial\}$.*

As can be seen from the previous results, the differences between exchangeable and weakly exchangeable arrays are mostly a matter of notation. That is why, from now on, we mostly focus on the exchangeable case; the corresponding statements for the weakly exchangeable case can be derived easily.

6.2 RELATION TO EXCHANGEABLE GRAPHS AND GRAPH LIMITS

The above construction is a straightforward generalisation of the graph limit construction from the theory of dense random graphs, which we recall briefly.

A (vertex) *exchangeable random graph* is a random graph G with countably many vertices labelled by $\mathbb{N}$ whose distribution is invariant under permutations of the labels. By considering the adjacency matrix $(G_{ij})_{i,j\in\mathbb{N}}$ of this graph, it can be viewed as a $\{0,1\}$-valued weakly exchangeable array whose diagonal entries are 0.

Graph limits were introduced by Lovász and Szegedy [143] (see also Borgs & Chayes [37]) while studying sequences of dense graphs. They encode the limiting density of finite subgraphs in an infinite graph. Formally, let $\mathcal{G}_n$ be the set of all graphs with n vertices labelled by $[n]$. For $m \geq n$ and $F \in \mathcal{G}_n$ and $G \in \mathcal{G}_m$, let $\mathrm{Hom}(F,G)$ be the number of injections $\psi\colon [n] \to [m]$ such that $G_{\psi(i)\psi(j)} = F_{ij}$ for all $i,j \in [n]$. Then, for every infinite graph G with vertices labelled by $\mathbb{N}$, one can define the "density of F in G"

$$t(F,G) = \lim_{m\to\infty} \frac{\mathrm{Hom}(F,G|_{[m]})}{(m)_n}, \qquad F \in \mathcal{G}_n. \tag{6.12}$$

It can be checked easily that $t(\cdot,G)$, restricted to $\mathcal{G}_n$, if it exists, is a probability measure on $\mathcal{G}_n$. This probability measure, in fact, coincides with the measure $t^{G,n}$ that was introduced in (6.11), when graphs are identified when their adjacency matrices.

By construction, every $t(\cdot,G)$ is invariant under action of Σ,

$$t(F^\pi,G) = t(F,G), \quad \text{for every } F \in \mathcal{G}_n,\ \pi \in \Sigma_n. \tag{6.13}$$

Similarly, the following consistency relation, corresponding to (6.10) above, holds:

$$t(F,G) = \sum_{\bar{F}\in\mathcal{G}_m : \bar{F}|_{[n]}=F} t(\bar{F},G). \tag{6.14}$$

That means that $(t(F,G))_{F\in\cup_n\mathcal{G}_n}$, if it exists for every $F \in \cup_n\mathcal{G}_n$, can be viewed (again in the sense of Kolmogorov's extension theorem) as a distribution of a random graph, which must be exchangeable due to (6.13). This distribution corresponds to $|y|$ of the previous section.

6.3 DYNAMICS OF EXCHANGEABLE ARRAYS

We now turn to the main goal of this chapter, the investigation of processes $X = (X(t))_{t\in T}$ taking values in the space $\mathbb{S}$ of two-dimensional S-valued arrays. Here, T denotes the set of times which can be both discrete, $T = \mathbb{N}_0$, or continuous $T = [0,\infty)$.

In the continuous-time case, we assume that the sample paths of X are càdlàg. Since we endowed $\mathbb{S}$ with the product topology, this is the case iff every restriction $X|_{[n]}$ has càdlàg paths in $\mathbb{S}_n$, or equivalently, $t \mapsto X_{ij}(t)$ is càdlàg for every $i,j \in \mathbb{N}$. We write, $D(S)$ for the space of all càdlàg functions from T to S, endowed with the usual Skorokhod topology. The previous reasoning implies that $D(\mathbb{S}) = (D(S))^{\mathbb{N}^2}$.

By convention, every function on T is càdlàg in the discrete-time case. This allows us to use the adjective 'càdlàg' without specifying which case we consider.

A $\mathbb{S}$-valued process X is called *exchangeable*, if

$$X^{\pi} := (X^{\pi}(t))_{t\in T} \overset{\text{law}}{=} X, \qquad \text{for every } \boldsymbol{\pi} \in \Sigma^2. \tag{6.15}$$

Equivalently, viewing X as an array of functions $(t \mapsto X_{ij}(t))_{i,j\in\mathbb{N}}$, it is often useful to regard X as an exchangeable $D(S)$-valued array. Corresponding to this point of view, we define an exchangeable σ-field, $\mathcal{E}_X$, associated to the whole process,

$$\mathcal{E}_X = \{X^{-1}(A) : A \in \mathcal{E}_{D(S)}\}, \tag{6.16}$$

where $\mathcal{E}_{D(S)}$ is defined as $\mathcal{E}_S$ with $D(S)$ playing the role of S.

The process X is a *Markov process* when the Markov property holds, that is the past $(X(s))_{s\leq t}$ and the future $(X(s))_{s\geq t}$ are conditionally independent given the present $X(t)$ for all $t \in T$. The following proposition gives criteria implying the exchangeability of a Markov process. Its straightforward proof is left to the reader.

Proposition 6.3.1. *Let X be an $\mathbb{S}$-valued Markov process with transition probability kernel*

$$p_{s,t}(x, A) := P[X(t) \in A \mid X(s) = x], \qquad s < t \in T, A \in \mathcal{B}(\mathbb{S}). \tag{6.17}$$

Then, X is exchangeable if

(a) *its initial state $X(0)$ is an S-valued exchangeable array, that is*

$$X(0)^{\pi} \overset{\text{law}}{=} X(0). \tag{6.18}$$

(b) *its transition kernels are invariant under action of Σ^2, that is for every $\boldsymbol{\pi} \in \Sigma^2$, $s < t \in T$, $x \in \mathbb{S}$, and $A \in \mathcal{B}(\mathbb{S})$*

$$p_{s,t}(x^{\pi}, A^{\pi}) = p_{s,t}(x, A). \tag{6.19}$$

For convenience, we mostly omit "$\mathbb{S}$-valued" from the terminology and say, e.g., "exchangeable Markov process" instead of "$\mathbb{S}$-valued exchangeable Markov process".

We now study how exchangeable Markov processes interact with the "projection" operation $\mathbb{S} \ni y \mapsto |y| \in \mathcal{D}^{\star}_S$, cf. Remark 6.1.1. Our first result implies that the projection of $X(t)$ is in $\mathcal{D}_S$, a.s., simultaneously for all $t \in T$, that is one can, a.s., project the process X on the space $\mathcal{D}_S$ of (the distributions of) dissociated exchangeable arrays, cf. Proposition 6.1.2. Remark that Markov property is not assumed.

Theorem 6.3.1. *Let X be an exchangeable process with càdlàg sample paths. Then, P-a.s., $|X(t)| \in \mathcal{D}_S$ for all $t \in T$.*

Proof. In the discrete-time case, it suffices to observe that $X(t)$ is an exchangeable S-valued array for every $t \in \mathbb{N}_0$. By Proposition 6.1.2, $|X(t)| \in \mathcal{D}_S$, P-a.s., and the claim follows, since $\mathbb{N}_0$ is countable.

In the continuous-time case, we view X as a $D(S)$-valued exchangeable array, cf. the remark below (6.15), and assume that this array is dissociated first. Using Proposition 6.1.2 with $D(S)$ in place of S, recalling that $|Y|$ there denotes the sequence of limits $(t^{Y,n})_{n\in\mathbb{N}}$, we see that for every $n \in \mathbb{N}$, the sequence $t^{X,n}_m$ of probability measures on $D(\mathbb{S}_n)$ converges weakly as

$m \to \infty$ P-a.s. to some $t^{X,n} \in \mathcal{P}(D(\mathbb{S}_n))$. Moreover, since we assume that X is dissociated, $t^{X,n}$ is a.s. deterministic and coincides with the distribution of $X|_{[n]}$, by Proposition 6.1.2(b).

Let J_n^X be the (deterministic) set of times defined by

$$J_n^X = \{t \in T : t^{X,n}(\{x \in D(\mathbb{S}_n) : t \text{ is a jump point of } x\}) > 0\}. \tag{6.20}$$

By the general theory of probability measures on Skorokhod spaces, see Chapter 15 in [28], J_n^X is at most countable. Therefore, using the same argument as in the discrete case, P-a.s., $|X(t)| \in \mathcal{D}_S$ for all $t \in \cup_n J_n^X$.

For $t \in T \setminus \cup_n J_n^X$, the coordinate projections $\phi_t \colon D(\mathbb{S}_n) \ni x \mapsto x(t) \in \mathbb{S}_n$ are $t^{X,n}$-a.s. continuous. By [28, Theorem 5.1], the weak convergence of $t_m^{X,n}$ then implies the existence of the weak limit $t^{X(t),n} := \phi_t \circ t^{X,n} = \lim_{m\to\infty} \phi_t \circ t_m^{X,n} = \lim_{m\to\infty} t_m^{X(t),n}$. The limit measures $t^{X(t),n} \in \mathcal{P}(\mathbb{S}_n)$ are consistent and dissociated, as $t_m^{X,n}$ are, and thus determine a probability measure $|X(t)| \in \mathcal{D}_S$, P-a.s., simultaneously for all $t \in T \setminus \cup_n J_n^X$.

The last two paragraphs together imply that for a dissociated X, $P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T] = 1$.

A general exchangeable càdlàg process X can be written as a mixture of dissociated processes by conditioning on $\mathcal{E}_X$, by Proposition 6.1.1. Therefore,

$$\begin{aligned} &P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T] \\ &\quad = \int_\Omega P[|X(t)| \in \mathcal{D}_S \text{ for all } t \in T \mid \mathcal{E}_X](\omega) P(\delta\omega). \end{aligned} \tag{6.21}$$

Under $P[\cdot \mid \mathcal{E}_X]$, the law of X is dissociated, and thus the integrand equals 1, a.s., by the previous paragraph. This completes the proof. □

From Proposition 6.1.2, we know that $|X(t)|$ is a regular conditional distribution of $X(t)$ given its own exchangeable σ-field $\mathcal{E}_{X(t)}$. In general, however, $\mathcal{E}_{X(t)}$ does not need to agree with $\mathcal{E}_X$. We now show that $|X(t)|$ is also a regular conditional distribution of $X(t)$ given $\mathcal{E}_X$.

Lemma 6.3.1. (a) *For every $t \in T$,*

$$\mathcal{E}_{X(t)} \subset \mathcal{E}_X.$$

(b) *Let α^X be the regular conditional distribution of X given $\mathcal{E}_X$. Then, P-a.s.,*

$$\alpha^X(\omega, X(t) \in \cdot) = |X(t)|(\omega, \cdot). \tag{6.22}$$

or, equivalently, denoting by ϕ_t the projection $D(\mathbb{S}) \ni x \mapsto x(t) \in \mathbb{S}$,

$$\phi_t \circ \alpha^X = |X(t)|. \tag{6.23}$$

Proof. (a) Let $B \in \mathcal{E}_S$. Then $\phi_t^{-1}(B) \in \mathcal{E}_{D(\mathbb{S})}$, and thus $X^{-1}(\phi_t^{-1}(B)) \in \mathcal{E}_X$. In addition,

$$\begin{aligned} X^{-1}(\phi_t^{-1}(B)) &= \{\omega \in \Omega : X(\omega) \in \phi_t^{-1}(B)\} \\ &= \{\omega \in \Omega : (\phi_t \circ X)(\omega) \in B\} \\ &= \{\omega \in \Omega : X(t)(\omega) \in B\} = X(t)^{-1}(B). \end{aligned} \tag{6.24}$$

Since, by definition, $\mathcal{E}_{X(t)} = \{X(t)^{-1}(B) : B \in \mathcal{E}_S\}$, it follows that $\mathcal{E}_{X(t)} \subset \mathcal{E}_X$, as claimed.

(b) Heuristically, the proof uses the fact that $|X(t)|$ is a dissociated distribution, and dissociated distributions are extremal in the set of all exchangeable distributions.

By properties of regular conditional distributions, for every $C \in \mathcal{E}_X$, and every bounded measurable $f\colon \mathbb{S} \to \mathbb{R}$,

$$E[\mathbf{1}_C f(X(t))] = \int_\Omega \mathbf{1}_C(\omega)\alpha^X(\omega, f \circ \phi_t) P(\delta\omega). \tag{6.25}$$

By conditioning on $\mathcal{E}_{X(t)}$, we obtain

$$(6.25) = \int_\Omega P(\delta\omega') \int_\Omega P(\delta\omega \mid \mathcal{E}_{X(t)})(\omega')\mathbf{1}_C(\omega)\alpha^X(\omega, f \circ \phi_t). \tag{6.26}$$

Choosing $C \in \mathcal{E}_{X(t)} \subset \mathcal{E}_X$ and using that $\mathbf{1}_C(\omega) = \mathbf{1}_C(\omega')$, $P(\cdot \mid \mathcal{E}_{X(t)})(\omega')$-a.s., in this case, we get

$$(6.25) = \int_\Omega P(\delta\omega')\mathbf{1}_C(\omega') \int_\Omega P(\delta\omega \mid \mathcal{E}_{X(t)})(\omega')\alpha^X(\omega, f \circ \phi_t), \tag{6.27}$$

Observe that, as function of ω', the inner integral is $\mathcal{E}_{X(t)}$ measurable. Therefore,

$$\int_\Omega P(\delta\omega \mid \mathcal{E}_{X(t)})(\phi_t \circ \alpha^X)(\omega) \tag{6.28}$$

is a version of regular conditional distribution of $X(t)$ given $\mathcal{E}_{X(t)}$, that is it equals $|X(t)|$, P-a.s. However, $|X(t)|$ is dissociated, and thus extremal in the set of all exchangeable probability distributions. Therefore, necessarily, $(\phi_t \circ \alpha^X)(\omega) = |X(t)|(\omega)$ must hold true for $P(\cdot \mid \mathcal{E}_{X(t)})$-a.e. ω. This then implies that $\phi_t \circ \alpha^X = |X(t)|$, P-a.s., as claimed. $\square$

Theorem 6.3.2. *Let X be an exchangeable process with càdlàg sample paths. Then, the projection $|X| = (|X(t)|)_{t\in T}$ has P-a.s. càdlàg sample paths.*

Theorem 6.3.3. *Let X be an exchangeable Markov process with càdlàg sample paths. Then, $|X|$ is a $\mathcal{D}^\star_S$-valued Markov process with a.s. càdlàg sample paths.*

6.4 JUMPS OF DISCRETE-TIME MARKOV PROCESSES

In this and the next section, we study in detail the structure of the jumps of time-homogeneous exchangeable Markov processes. We first consider processes in discrete time, where the situation is rather simple.

Lemma 6.4.1. *Let X be an exchangeable Markov process in discrete time. Then, the array $J_{ij}(t) = \mathbf{1}\{X_{ij}(t-1) \neq X_{ij}(t)\}$ encoding its jumps at time $t \geq 1$ is also exchangeable. As consequence, only the following two possibilities occur a.s.*

- *X is constant at t, that is $X_{ij}(t-1) = X_{ij}(t)$ for all $(i,j) \in \mathbb{N}^2$.*
- *There is a positive proportion of entries which jump, that is*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\leq i,j\leq n} \mathbf{1}\{X_{ij}(t-1) \neq X_{ij}(t)\} > 0. \tag{6.29}$$

6.5 RESTRICTIONS OF MARKOV EXCHANGEABLE PROCESSES ARE NOT MARKOV

If one is interested not only in the occurrence of jumps, but also in their "sizes", this argument can be pushed even further, similarly to [57]. For $t \geq 1$, consider S^2-valued array $Z_{ij} := (X_{ij}(t-1), X_{ij}(t))$, which is again exchangeable. By Proposition 6.1.2 (with S^2 in place of S), for every $n \in \mathbb{N}$, the limit $t^{Z,n} \in \mathcal{P}(\mathbb{S}_n^2)$ exists a.s.

The measure $t^{Z,n}$ can be used to construct a new Markov transition kernel q_n on $\mathbb{S}^n$, by disintegrating $t^{Z,n}$ with respect to its first marginal $t^{X(t-1),n}$,

$$t^{Z,n}(\delta y_1, \delta y_2) = t^{X(t-1),n}(\delta y_1) q^n_{t-1,t}(y_1, \delta y_2), \tag{6.30}$$

or, in the case when S is finite, simply by defining

$$q^n_{t-1,t}(y_1, y_2) = \frac{t^{Z,n}(\{(y_1, y_2)\})}{t^{X(t-1),n}(\{y_1\})}, \qquad y_1, y_2 \in \mathbb{S}_n, \tag{6.31}$$

(and $q^n_{t-1,t}(y_1, y_2) = \delta_{y_1,y_2}$ in the case when $t^{X(t-1),n}(\{y_1\}) = 0$). Since, by Proposition 6.1.2, $t^{X,n}$ agrees with the distribution of $X|_{[n]}$ given $\mathcal{E}_X$, it is tempting to interpret the kernels q_n as transition kernels of $X|_{[n]}$ (at least conditionally on $\mathcal{E}_X$), as is done in [57]: Proposition 4.8 of [57] contains, among others, the following claim (stated in the notation of the present chapter):

> Let $X = (X_t)_{t \in T}$ be a time-homogeneous exchangeable Markov process, with T being finite. Conditioned on $\mathcal{E}_X$, X is dissociated, and, moreover, for every $n \in \mathbb{N}$, the restriction $X|_{[n]}$ of X to $\mathbb{S}_n$ is (conditionally) a time-inhomogeneous Markov chain with transition probabilities $q^n_{t-1,t}$.

We now provide a counterexample for a part of this claim, namely that $X|_{[n]}$ is (conditionally) Markov. We will also see that the transition kernel of $X|_{[n]}$ is not q^n.

Example 6.5.1. *We work in the setting of exchangeable random graphs, similarly as in [57]. That is $X_{ij}(t)$ denotes the adjacency matrix of a random exchangeable graph, which can thus be viewed as $\{0,1\}$-valued weakly exchangeable array with zeros on the diagonal. We fix $T = \{0, 1, \dots, N\}$ for a large N.*

To construct the process, let ξ_i, $i \in \mathbb{N}$, be i.i.d. Bernoulli($\frac{1}{2}$) random variables. In the initial configuration $X(0)$, we draw an edge between vertices $i \neq j$ (i.e., we set $X_{ij}(0) = 1$) with probability $p_{ij}(\xi)$, where

$$p_{ij}(\xi) = \begin{cases} \frac{1}{4}, & \text{if } \xi_i = \xi_j = 0, \\ \frac{1}{2}, & \text{if } \xi_i \neq \xi_j, \\ \frac{3}{4}, & \text{if } \xi_i = \xi_j = 1. \end{cases} \tag{6.32}$$

All edges are drawn independently.

To define the dynamics, for every $x \in \mathbb{S}$, we define

$$\xi_i(x) = \mathbf{1}\Big\{ \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} x_{ij} > \tfrac{1}{2} \Big\}. \tag{6.33}$$

Given the configuration of X at time t, we construct $X(t+1)$ as follows

- *If $\xi_i(X(t)) = \xi_j(X(t)) = 0$, then X_{ij} does not change, that is $X_{ij}(t+1) = X_{ij}(t)$.*
- *Otherwise, X_{ij} is refreshed according to $p_{ij}(X(t))$, that is $X_{ij}(t)$ is a Bernoulli($p_{ij}(\xi(X(t))$) random variable, chosen independently of all other $X_{ij}(t)$'s.*

It is easy to see that the process X is weakly exchangeable. And, by construction, it is obviously Markov. In addition, the law of large numbers implies that $\xi_i(X(0)) = \xi_i$ a.s., and thus $X(1)$, and inductively also $X(t)$, $t \geq 1$, have the same distribution as $X(0)$.

The exchangeable σ-field $\mathcal{E}_X$ is P-trivial in this example, since X is dissociated by construction. Hence, conditioning on $\mathcal{E}_X$ does not have any effect.

On the other hand, the functions $\xi_i(X(t))$ cannot be determined from any finite restriction $X(t)|_{[n]}$. That is, ξ's are "hidden variables" for the restriction $X|_{[n]}$, and while conditionally on ξ, $X|_{[n]}$ is Markov, it is not Markov unconditionally.

To prove this, fix $n = 2$, that is consider only the state of the edge connecting the vertices 1 and 2. Then, by an easy computation taking into account all possible values of ξ_1 and ξ_2, we obtain that $P(X_{12}(t+1) = 1 \mid X_{12}(t) = 1) = \frac{21}{32}$. On the other hand, $P(X_{12}(N) = 1 \mid X_{12}(t) = 1, \forall t < N)$ can be made arbitrarily close to one by choosing N large, because if we know that $X_{12}(t) = 1$ for all $t < N$, then very likely $\xi_1 = \xi_2 = 0$ and thus X_{12} never flips:

$$\begin{aligned} P(X_{12}(N) = 1 \mid X_{12}(t) = 1, \forall t < N) &= \frac{P(X_{12}(N) = 1, \forall t \leq N)}{P(X_{12}(N) = 1, \forall t < N)} \\ &= \frac{\frac{1}{4}\cdot\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot(\frac{1}{2})^N + \frac{1}{4}\cdot(\frac{3}{4})^N}{\frac{1}{4}\cdot\frac{1}{4}\cdot 1 + \frac{1}{2}\cdot(\frac{1}{2})^{N-1} + \frac{1}{4}\cdot(\frac{3}{4})^{N-1}} \xrightarrow{N\to\infty} 1. \end{aligned} \tag{6.34}$$

This implies that X_{12} is not Markov.

Remark 6.5.1. (a) *On the technical level, the problem with the argument in [57] is: The relation (14) therein, which gives certain consistency for the kernels q_n, does not hold true, in general. This can hinder the Markov property of the finite restrictions as shown in Example 6.5.1.*

(b) *However, in Section 6.7 (see Theorem 6.7.1), we show that under the additional assumption that the "global" Markov process X has the Feller property (cf., Definition 6.7.1), all the "local" restrictions $X|_{[n]}$ are indeed Markov (and Feller). See also Remark 6.7.1.*

6.6 JUMPS OF CONTINUOUS-TIME MARKOV PROCESSES

We now study exchangeable Markov processes in continuous time. Similarly as in discrete time (see Lemma 6.4.1), we describe the possible jumps of this process. The structure here is richer, because the process is indexed by an uncountable set of times. So, certain events which have probability 0 in the discrete settings can occur.

Theorem 6.6.1. *Let X be exchangeable Markov process with càdlàg paths in continuous time, and let $J \subset (0, \infty)$ be the (random) set of times when $t \mapsto X_t$ is discontinuous. Then, a.s., J can be written as a disjoint union $J = J^1 \,\dot\cup\, J^2 \,\dot\cup\, J^3$, where*

- *J^1 is the set of times, where a positive proportion of entries of X jumps,*

$$J^1 := \left\{ t > 0 : \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\leq i,j\leq n} \mathbf{1}\{X_{ij}^-(t) \neq X_{ij}(t)\} > 0 \right\}, \tag{6.35}$$

- J^2 *is the set of times, where a positive proportion of entries in one row or column of X jumps,* $J^2 = J^{2,c} \,\dot\cup\, J^{2,r}$ *with*

$$J^{2,r} = \Big\{ t > 0 : \exists! i \in \mathbb{N} \text{ s.t. } X^-_{i'j}(t) = X_{i'j}(t), \forall i' \neq i, j \in \mathbb{N}, \\ \text{and } \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}\{X^-_{ij}(t) \neq X_{ij}(t)\} > 0 \Big\}, \tag{6.36}$$

$$J^{2,c} = \Big\{ t > 0 : \exists! j \in \mathbb{N} \text{ s.t. } X^-_{ij'}(t) = X_{ij'}(t), \forall j' \neq j, i \in \mathbb{N}, \\ \text{and } \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{X^-_{ij}(t) \neq X_{ij}(t)\} > 0 \Big\}, \tag{6.37}$$

- J^3 *is the set of times, where a unique entry jumps,*

$$J^3 = \{ t > 0 : \exists!\, i, j \in \mathbb{N} \text{ s.t. } X_{ij} \text{ is discontinuous at } t, \text{ and} \\ X_{i'j'} \text{ is continuous at } t, \forall (i', j') \neq (i, j) \}. \tag{6.38}$$

From Theorem 6.6.1, we deduce the following claim about the discontinuities of the projection $|X|$.

Corollary 6.6.1. *Let $J^{|X|}$ be the set of times when $t \mapsto |X(t)|$ is discontinuous. Then, $J^{|X|} \subset J^1$, where J^1 is as in* Theorem 6.6.1.

The inclusion in the previous theorem might be strict. As an example, consider the process started from $X_{ij}(0)$ being i.i.d. Bernoulli($\frac{1}{2}$), where all entries are refreshed simultaneously by an independent i.i.d. Bernoulli($\frac{1}{2}$) array at times of jump of a standard Poisson process N_t. In this case, for every $t \geq 0$, $|X(t)|$ is the distribution of the i.i.d. Bernoulli($\frac{1}{2}$) array, that is $J^{|X|} = \emptyset$. On the other hand, J^1 agrees with the set of jumps of N_t.

6.7 THE FELLER PROPERTY

In the last part of this chapter, we discuss the conditions under which exchangeable S-valued Markov processes in continuous time have the Feller property. Recall the following.

Definition 6.7.1. *A time-homogeneous S-valued Markov process with transition kernels $p_t(\cdot, \cdot)$ is called* Feller *if*

(a) *For every $g \in C_b(S)$, $t \geq 0$ and $y \in S$, the map $x \mapsto \int g(y) p_t(x, \delta y)$ is continuous.*

(b) *For every $x \in S$ and $g \in C_b(S)$,*

$$\lim_{t \downarrow 0} \int g(y) p_t(x, \delta y) = g(x). \tag{6.39}$$

It is easy to construct exchangeable Markov processes that are not Feller. E.g., the process considered in Example 6.5.1 does not satisfy (a) of the Feller property. To see this, take $g(y) = y_{12}$, $y \in \mathbb{S}$, and observe that for every $t > 0$ there is $\varepsilon_t > 0$ such that if $x_{12} = 1$, then

$$\int g(y) p_t(x, \delta y) \begin{cases} = 1, & \text{if } \xi_1(x) = 0 \text{ and } \xi_2(x) = 0, \\ < 1 - \varepsilon_t, & \text{otherwise.} \end{cases} \tag{6.40}$$

Inspecting, the definition (6.33) of $\xi_i(x)$, it is easy to see that it is not continuous function of x, and thus $X(t)$ is not Feller.

This example indicates one possibility of how the Feller property can be violated by exchangeable Markov processes: If the transition kernel depends on "non-local exchangeable quantities", then the process is not Feller. We now show that this is essentially the only mechanism, how the Feller property can be violated.

The following definition imposes a very strong "locality" of the distribution of X.

Definition 6.7.2. *An exchangeable Markov process X is called consistent if its every restriction $X|_{[n]}$ to $\mathbb{S}_n$ is Markov with respect to its own natural filtration.*

Theorem 6.7.1. *For a time-homogeneous exchangeable Markov process X, the following are equivalent:*

(i) *X is consistent and every $X|_{[n]}$ is a Feller process on $\mathbb{S}_n$.*

(ii) *X is Feller.*

Remark 6.7.1. *If S is finite, then $\mathbb{S}_n$ is finite as well. Every càdlàg Markov process on a finite state space is Feller. Therefore, in this case, the consistency of X is equivalent to Feller property. This was proved in the exchangeable random graph case in [58].*

6.8 DISCUSSION AND OUTLOOK

LIMITATIONS. Vertex exchangeability might be a problematic assumption for some applications:

- All vertices might not be exchangeable.
- Vertex-exchangeability implies that the network is either *dense* (or empty), a.s. However, the networks in some applications are known to be *sparse*.

OUTLOOK. To construct theories of IPS on evolving networks, one might use the following ingredients:

- Combine *random graphs and complex networks* (e.g., Durrett [81] and van der Hofstad [188]) with *stochastic processes in evolving random environments* (see, e.g., Andres *et al.* [8], Athreya *et al.* [15], Birkner *et al.* [30], and Peres *et al.* [169]).
- Use the stochastic limiting structures for large graphs (see, e.g., Borgs & Chayes [37] and Crane [56]) to describe the large IPS on evolving networks.
- Encode the limiting and prelimiting structures as stochastic processes on evolving metric measure spaces. A somewhat related idea is being explored in an analytic context by Kopfer & Sturm [130].

OPEN PROBLEMS:

- Study your favorite finite IPS on your favorite evolving network. See, e.g., Jacob *et al.* [117] and Jacob & Mörters [118] for some rare rigorous examples.
  - Study (scaling) limits/universality in this context.
  - Are exchangeable graph/particle models (scaling) limits of any finite IPS on evolving networks?
- Characterize Markovian network dynamics of *sparse* (e.g., Caron & Fox [51], Crane & Dempsey [59], and Janson [119]) random networks.
- Study adaptive (a.k.a. coevolving) models, i.e., allow for interactions between the particle states and network evolution, see, e.g., Basu & Sly [22] and Chatterjee *et al.* [52], for some rare rigorous analyses.
- Statistical inference, estimation, uncertainty quantification for stochastic processes on evolving networks. E.g.,
  - Infer the network geometry from the behavior of an IPS on it.

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